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TRANSREAL NEWTONIAN PHYSICS OPERATES AT SINGULARITIES

A FÍSICA NEWTONIANA TRANSREAL OPERA EM SINGULARIDADES*

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Abstract: Sir Isaac Newton, writing in Latin, defined his celebrated laws of motion verbally. When the laws of motion are read as relating to his arithmetic (the Arithmetica Universalis) and his calculus (the method of fluxions), division by zero is undefined so his physics fails at mathematical singularities. The situation is unchanged in modern real arithmetic and real calculus: division by zero is undefined so both Newtonian Physics and its modern developments fail at mathematical singularities. However, when Newton's text is read as relating to transreal arithmetic and transreal calculus, division by zero is defined and we show that the resulting Transreal Newtonian Physics does operate at mathematical singularities. We hold out the hope that the whole of modern physics may be similarly extended. We use the new physics to predict a convection current at the singularity in a black hole and suggest experiments in astronomy and high energy physics that might confirm or rebut our predictions. Thus our exegesis of Newton's text extends mathematical physics and may, in future, extend experimental physics.

Keywords: Transreal numbers; Non-finite force; Newton's laws of motion.

Resumo: Sir Isaac Newton, escrevendo em latim, definiu verbalmente suas célebres leis do movimento. Quando as leis do movimento são lidas relativas à sua aritmética (a *Arithmetica Universalis*) e ao seu cálculo (o método das fluxões), a divisão por zero não é definida. Assim, a física de Newton falha em singularidades matemáticas. Esta situação se mantém na aritmética e cálculo real modernos: a divisão por zero é indefinida e, por isso, tanto a física newtoniana quanto seus desenvolvimentos modernos falham em singularidades matemáticas. No entanto, quando o texto de Newton é lido relativo à aritmética transreal e ao cálculo transreal, a divisão por zero é bem definida. Desta forma, mostramos que a Física Newtoniana Transreal resultante opera em singularidades matemáticas. Esperamos que toda a física moderna possa ser estendida

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de forma análoga. Usamos esta nova física para prever uma corrente de convecção na singularidade em um buraco negro e sugerimos experimentos em astronomia e física de alta energia que podem confirmar ou refutar nossas previsões. Portanto, nossa exegese do texto de Newton estende a física matemática e poderá, no futuro, estender a física experimental.

Palavras-chave: Números transreais; Força não-finita; Leis do movimento de Newton.

1. Introdução

Newton worked substantially in Latin but his celebrated Principia (NEWTON, 1760) and many of his mathematical papers are available in English translation (COHEN and WHITMAN, 1999) (WHITESIDE, 1974). We know what arithmetical techniques Newton used because he left many pages of manuscript computation and his lecture notes on arithmetic (algebra) survive. His arithmetic became known as the Arithmetica Universalis, see (WHITESIDE, 1974, v. 5) and, for our purposes, especially pages 61-107. The Arithmetica Universalis differs from modern, real arithmetic, not in terms of numerical results but in algebraic scope, as we show in the Section Arithmetica Universalis, but it shares the property that division by zero is undefined. We suppose, therefore, that Newton could not divide by zero. This view is supported, to an extent, by Newton's early notes on indivisibles in which he records division with a zero that is not the zero of modern, real arithmetic but is an abstract symbol in an inchoate account of infinitesimal numbers. Hence Newton could find limits of some functions whose denominator tends *asymptotically* to zero but could not divide by an *exact* zero. Exact division by zero is undefined in Newton's fluxions and in both modern real and complex analysis. Consequently, we suppose, Newton read his three laws of motion as failing on division by zero, as many modern readers continue to do. Transreal and transcomplex arithmetic are developments of Computer Science that are now being normalised in Mathematics (ANDERSON and REIS, 2014) (GOMIDE and REIS, 2013) (REIS, 2014) (REIS and ANDERSON, 2014a, 2014b, 2015) (REIS et al, 2013). They define division in terms of operations on the lexical reciprocal. This lexical definition contains the usual definition of division, as multiplication by the multiplicative inverse, but also defines division by zero. Consequently transreal and transcomplex arithmetic are supersets of, respectively, real and complex arithmetic. There is a machine proof that transreal arithmetic is consistent if real arithmetic is (ANDERSON et al, 2007). The consistency of the sets of transreal and transcomplex numbers is also ensured by their constructions from the real and complex numbers (REIS and ANDERSON, 2014b). We show, in the Sections Laws of Motion and Discussion, how to express Newton's three laws of motion in transreal arithmetic so that they are defined on division by zero.

Transreal arithmetic uses a subset of the algorithms of real arithmetic so the general reader will be able to follow any computation in transreal arithmetic but will have little chance of deriving a valid computation until the axioms (ANDERSON et al, 2007) or algorithms (ANDERSON, 2011) of transreal arithmetic have been properly learned. For the reader's convenience the algorithms are given, with permission, in a tutorial in the Section Transreal Arithmetic.

An important property of transreal arithmetic is that it is total, which means that every operation of the arithmetic can be applied to any numbers with the result being a number. By contrast the Arithmetica Universalis and real arithmetic are partial because division by zero is not defined. Newton defined his laws of motion verbally. When his words are read as relating to the Arithmetica Universalis or to real arithmetic, the arithmetical versions of the laws fail on division by zero. Consequently Newtonian Physics fails at singularities. By contrast when the verbal laws are read as relating to transreal arithmetic, the laws succeed on division by zero because transreal arithmetic is total. When we examine the laws, in detail, we find that, for Newtonian particles of a given mass, the transreal laws provide a coherent account of force and acceleration so we say that Transreal Newtonian Physics operates at singularities. However, the transreal laws give a more complicated account of mass at a singularity: the mass is both a given, finite number and nullity. In the Section Discussion we resolve this dialethia, or true contradiction, by noting that the mass is a hidden variable in the singularity but is observable, by its gravitational effects, outside the singularity.

The plan of the remainder of the paper is to summarise the salient features of transreal arithmetic in the Section Transreal Arithmetic. In the Section Arithmetica Universalis we show, very briefly, how the Arithmetica Universalis overlaps with, but is different from, both real and transreal arithmetic. In the Section Laws of Motion we deal with Newton's laws in two steps. Firstly we define how each of Newton's three laws of motion is to be read in transreal arithmetic. Secondly we perform computations on the second law to show that the behaviour of Newtonian particles is conceptually coherent. We then generalise this transarithmetical reading to versions of the laws stated in terms of the transdifferential. In the Section Discussion we briefly consider

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how the new reading of Newton's laws might be applied at mathematical singularities and how the totality of the resulting mathematical analyses might be exploited in computer programs and novel computer hardware. In the Section Conclusion we conclude with a statement of the main, original contributions of the paper.

2. Transreal Arithmetic and Transreal Calculus

2.1 Relationship to Physics

Transreal arithmetic implies a topology (REIS and ANDERSON, 2015), Figure 1, that gives a definite, numerical value to the result of dividing any real number by zero. Infinity, ∞, is the unique number that results when a positive number is divided by zero; negative infinity, $-\infty$, is the unique number that results when a negative number is divided by zero; nullity, Φ, is the unique number that results when zero is divided by zero. Nullity is not ordered, all other transreal numbers are ordered. Infinity is the largest number and negative infinity is the smallest number. Any particular real number is finite; ∞ and $-\infty$ are infinite; Φ is non-finite. The infinite numbers are also non-finite. The real numbers, $\mathbb{R}$, together with the infinite numbers, $\{-\infty, \infty\}$, make up the extended-real numbers, $\mathbb{R}^E$; the real numbers, together with the non-finite numbers, $\{-\infty, \infty, \Phi\}$, make up the transreal numbers, $\mathbb{R}^T$. Transcomplex numbers, $\mathbb{C}^T$, are also available in (ANDERSON, 2011) and have been simplified in a construction from the set of complex numbers (REIS and ANDERSON, 2014b).

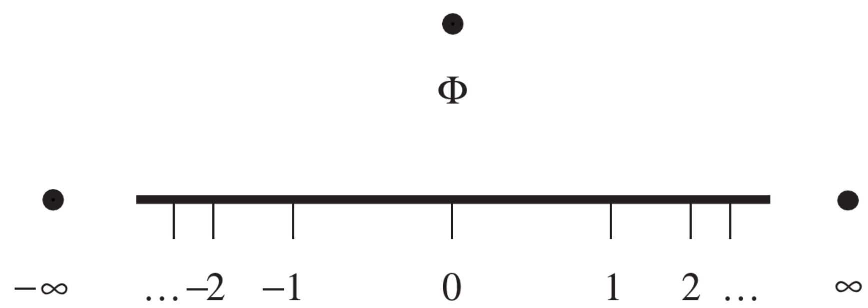

Figure 1. Transreal line

Anderson (2008) argues that infinite forces can influence the real numbered universe because negative infinity is less than any real number and positive infinity is greater than any real number whence there is a number intermediate between any real number and an infinity so that continuous, i.e. classical, motion can occur. But nullity is an unordered number, thus, there is no number intermediate between nullity and any other transreal number so no continuous, and no classical, motion can occur in response to a nullity force. The axiom that nullity forces have no influence on the extended-real universe is a hypothesis that links transreal arithmetic to Physics. We adopt this axiom here.

This axiom gains plausibility when we recognise that the point at nullity is topologically isolated from the extended-real numbers (REIS and ANDERSON, 2015) so that there is no component of a nullity force that lies inside the extended-real space. Therefore motions within an extended-real space are not perturbed by a nullity force.

In (REIS and ANDERSON, 2015) the interpretation that nullity has different topological properties from infinity is formalised. A further topology is imposed on the transreal numbers which allows infinite limits to exist in exactly the same way as in measure theory and integration. A very important consequence of this is that all of the results of real calculus extend to transreal calculus. In particular all of those results of mathematical physics that rely on the real derivative and real integral apply, without change, to the transreal numbers. The move to transreal calculus does, however, bring about an important change. In real calculus the infinity symbols, $-\infty$, ∞, refer to indefinite, real variables that grow without bound. In transreal calculus the infinity symbols, $-\infty$, ∞, refer to definite, transreal numbers that have a fixed value which can be computed using transreal arithmetic. Consequently transreal functions can be evaluated both as numerical values of the function and as limits of the function, where the limits exist. One may, therefore, obtain both the limit asymptotically close to a singularity and the function value exactly at the singularity. Hence one may test a function to determine whether or not it is continuous at the singularity. Thus armed, we have the mathematical tools needed to extend Newtonian physics so that it operates at singularities. We then use the Newtonian results as a basis for a bold conjecture that the event horizon of a black hole is not stable but roils.

Newton's works are open to interpretation but it is common practice to read Newton's Principia (NEWTON, 1760), translated in (COHEN and WHITMAN, 1999), as describing operations on point particles which have a fixed, finite, positive mass, m, such that $0 < m < \infty$. We adopt this conventional reading here.

2.2 Tutorial on Transreal Arithmetic

The reader might find it helpful to peruse this brief tutorial on transreal arithmetic which is reproduced and amended, with permission, from a conference paper (ANDERSON, 2011). The consistency of the set of transreal numbers is ensured by (ANDERSON et al, 2007) and their constructions from the real and complex numbers in (REIS and ANDERSON, 2014b).

Transreal numbers can be expressed as transreal fractions, n/d, of a real numerator, n, and a real denominator, d. Transreal fractions with a non-finite numerator or denominator simplify to this form. An improper, transreal fraction can be written with a negative denominator but it must be converted to a proper, transreal fraction, by carrying the sign up to the numerator, before applying any transreal, arithmetical operator. This can be done by multiplying both the numerator and denominator by minus one; it can be done by negating both the numerator and the denominator, using subtraction; and it can be done; lexically, by moving the minus sign from the denominator to the numerator. With $-d < 0$ we have:

$$\frac{n}{-d} = \frac{-1 \times n}{-1 \times (-d)} = \frac{-n}{-(-d)} = \frac{-n}{d}$$

Transreal infinity, ∞, is equal to any positive number divided by zero. Transreal minus-infinity, $-\infty$, is equal to any negative number divided by zero. Zero, 0, is equal to zero divided by any positive or negative number. Nullity, Φ, is equal to zero divided by zero. That is, with $k > 0$ we have:

$$\infty = \frac{1}{0} = \frac{k}{0} \qquad -\infty = \frac{-1}{0} = \frac{-k}{0} \qquad 0 = \frac{0}{1} = \frac{0}{k} = \frac{0}{-k} \qquad \Phi = \frac{0}{0}$$

The algorithms of real multiplication and division apply universally to proper, transreal fractions. That is, they apply without side conditions. In particular division by zero is allowed.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \tag{2. 1}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \tag{2. 2}$$

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Addition is more difficult than multiplication or division because it breaks into two cases: the addition of two signed infinities and the general case. Two infinities, in least terms, $1/0$ or $-1/0$, are added using the algorithm of real arithmetic for adding fractions with a common denominator:

$$\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{1+1}{0} = \frac{2}{0} = \frac{1}{0} = \infty \tag{2. 3}$$

$$\infty + (-\infty) = \frac{-1}{0} + \frac{1}{0} = \frac{-1+1}{0} = \frac{-1}{0} = \Phi \tag{2. 4}$$

$$(-\infty) + \infty = \frac{-1}{0} + \frac{1}{0} = \frac{-1+1}{0} = \frac{0}{0} = \Phi \tag{2. 5}$$

$$(-\infty) + (-\infty) = \frac{-1}{0} + \frac{-1}{0} = \frac{-1+(-1)}{0} = \frac{-2}{0} = \frac{-1}{0} = -\infty \tag{2. 6}$$

Addition of all other combinations of numbers uses the algorithm of real arithmetic for adding fractions without a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d + c \times b}{b \times d} \tag{2. 7}$$

Where both arguments (a/b) and (c/d) are finite, and $b = d$, the real algorithm for adding fractions with a common denominator may be used as a shorthand. The algorithm of real subtraction applies universally:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} \tag{2. 8}$$

Transreal addition and multiplication are associative:

$$(a + b) + c = a + (b + c)$$

$$(a \times b) \times c = a \times (b \times c)$$

Transreal addition and multiplication are commutative:

$$a + b = b + a$$

$$a \times b = b \times a$$

Transreal arithmetic is partially distributive:

$$a \times (b + c) = a \times b + a \times c$$

That is, the above equality holds, firstly, when $a \neq \pm\infty$; secondly when $a = \pm\infty$ and $b + c = 0$ or $b + c = \Phi$ or b, c have the same sign. Two numbers have the same sign when they are both positive or both negative or both zero or both nullity.

Despite the fact that transreal arithmetic has a partial distribution, it is still a total arithmetic, because both $a \times (b + c)$ and $a \times b + a \times c$ can be computed for any a, b, c.

2.3 Transreal Calculus

The calculus is one of the most important mathematical tools in today's world. Having its origins hundreds, perhaps thousands, of years ago. Without it, most current technological innovations would not exist. Calculus emerged and was developed by trying to understand nature. Primitive cultures conceived deities to control natural phenomena. Today we describe such phenomena with equations and computational models. We are like our ancestors; the difference lies only in our language and tools. We continue seeking for and giving explanations to the natural phenomena that surround us. Scientific research, mathematical research, philosophical thought, art and religious practice are faces of the same die. Namely, the seeking of knowledge. Transreal calculus is intended to extend the ordinary calculus to singularities. It is another lubricant for the human mind in its attempts to expand the horizons of mathematics and our understanding of the world. Transreal calculus allows the application of derivatives and integrals involving exact division by zero. The idea of dividing by zero, with transreals, originated in Computer Science, but these numbers are proving useful and their fields of application are expanding.

In (REIS and ANDERSON, 2015) the derivative and the integral are defined in transreal space. The derivative and integral are defined as limits and limits are defined in topological spaces. So the first step was to establish a topology in $\mathbb{R}^T$. In simple words, a topology is nothing more than a way to say when a point lies in the neighborhood of another. A topology for extended-reals is already known so all that had to be done was incorporate nullity into this topology. The topology of extended-reals is contingent on the order relation. As nullity is unordered, nullity is left as an isolated point. That is, there is a neighborhood of nullity that contains only itself. In this way:

A set is open (a neighborhood of its points) on $\mathbb{R}^T$ *if and only if it is composed of arbitrarily many unions of finitely many intersections of the following four kinds of interval:* (a, b), $[-\infty, b)$, $(a, \infty]$, $\{\Phi\}$ *where* $a, b \in \mathbb{R}$.

The details of this topology are given in the text cited above.

The second step followed directly by defining limit and continuity as usual in topological spaces. An important result from this is that transreal limit agrees with the real limit. That is, wherever real numbers occur in real limits, they occur identically in transreal limits and wherever infinities occur in real limits, they occur identically in transreal limits but as definite numbers, not only as symbols of divergence.

The third step was to define the derivative on $\mathbb{R}^T$. The way chosen was the intuitive way of defining the derivative as the slope of the tangent line. So the derivative at a real number is the usual derivative at that number. And if the limit of slopes of the tangent lines at x, when x tends to ∞ (or $-\infty$), is $L \in \mathbb{R}^T$ then the slope of the tangent line at ∞ (or $-\infty$) is L. Furthermore, it is nonsense to speak of a limit at an isolated point. Rather than accept the indeterminacy of the derivative at the isolated point, it is defined as Φ. Thus:

Let f *be a function on* $\mathbb{R}^T$. *If* $x \in \mathbb{R}$ *then* $f'_{\mathbb{R}^T}(x) = f'(x)$ *provided that this derivative exists. Further,* $f'_{\mathbb{R}^T}(-\infty) = \lim_{x \to -\infty} f'(x)$ *and* $f'_{\mathbb{R}^T}(\infty) = \lim_{x \to \infty} f'(x)$ *provided that these limits exist, and* $f'_{\mathbb{R}^T}(\Phi) = \Phi$.

This implies that the exponential is identically its own derivative with $e'(x) = e(x)$ so that the usual properties of this important function hold when extended to $\mathbb{R}^T$. The derivative of other important functions can be seen in a forthcoming paper[1]. Another observation is that the definition may not show it explicitly, but the derivative at $-\infty$ or ∞ also is, like the usual one, a rate of change of the function. It is shown that $f'_{\mathbb{R}^T}(-\infty) = \lim_{\substack{x\to-\infty \\ y\to-\infty}} \frac{f(y)-f(x)}{y-x}$ and $f'_{\mathbb{R}^T}(\infty) = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{f(y)-f(x)}{y-x}$ where these limits are defined in a suitable sense.

The fourth step was to define the integral on $\mathbb{R}^T$. The integral was defined in the usual way as infimum and supremum of integrals of step functions. The difference in the transreal case lies in the definitions of intervals, length of intervals and partitions of intervals. Let $a, b \in \mathbb{R}^T$. It was defined that $(a, b) = \{x \in \mathbb{R}^T;\ a < x < b\}$, $(a, b] = (a, b) \cup \{b\}$, $[a, b) = \{a\} \cup (a, b)$ and $[a, b] = \{a\} \cup (a, b) \cup \{b\}$. Notice that, in the usual way, $[a, b]$ would be defined as $\{x \in \mathbb{R}^T;\ a \leq x \leq b\}$, but then one would have $[a, \Phi] = \emptyset$. However, by the definition given, $[a, \Phi] = \{a, \Phi\}$. Concerning the length, if I is one of the intervals above then the length of I is: $|I| = 0$ if $I = \emptyset$, $|I| = k - k$ if $I = \{k\}$ for some $k \in \mathbb{R}^T$ and $|I| = \mathrm{b} - \mathrm{a}$ otherwise. Notice that, in the usual way, simply, $|I| = b - a$. But on $\mathbb{R}^T$, $|I|$ would then not be well defined. Because, for $a \in \mathbb{R}$, $|[a, \Phi)| = \Phi - a = \Phi$ and $|[a, a)| = a - a = 0$, but $[a, \Phi) = \{a\} = [a, a)$. Consequently there would be the absurdity $\Phi = |[a, \Phi)| = |[a, a)| = 0$. Concerning partitions, let $[a, b]$ be an interval. A set P is a partition of $[a, b]$ if there are $n \in \mathbb{N}$, $x_0, \ldots, x_n \in [a, b]$ such that $P = (x_0, \ldots, x_n)$ where $x_0 = a$, $x_n = b$ and, furthermore, if $n = 2$, $x_0 \leq x_1$ and if $n > 2$, $x_0 < x_1 < \cdots < x_{n-1} < x_n$. Notice that one could require, simply, $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$ but then the interval $[a, \Phi]$, for $a \in \mathbb{R}$ would not have any partitions because $a \leq \Phi$ does not hold. After that:

Let $a, b \in \mathbb{R}^T$ and $\varphi = \sum_{j=1}^{n} c_j X_{I_j}$ be a step function on [a,b] (X denoted the characteristic function). The integral in $\mathbb{R}^T$ of φ on $[a, b]$ was defined as

[1]REIS, T. S. dos; ANDERSON, J. A. D. W. Transreal Limits and Elementary Functions. *Transactions on Engineering Technologies - World Congress on Engineering and Computer Science 2014*, Springer. To appear.

$$_{\mathbb{R}^T}\int_a^b \varphi(x) = \sum_{\substack{j=1 \\ j;\, c_j \neq 0}}^{n} c_j |I_j|.$$

Now let $f\colon [a,b] \to \mathbb{R}^T$ *be a function. The integral in* $\mathbb{R}^T$ *of* f *on* $[a,b]$ *was defined as*

$$_{\mathbb{R}^T}\int_a^b f(x) = \inf\left\{ _{\mathbb{R}^T}\int_a^b \varphi(x)\,dx; \varphi \in \mathbf{S}([a,b]) \; and \; \varphi \not< f \right\}$$

provided that

$$\inf\left\{ _{\mathbb{R}^T}\int_a^b \varphi(x)\,; \varphi \in \mathbf{S}([a,b]) \; and \; \varphi \not< f \right\} = \sup\left\{ _{\mathbb{R}^T}\int_a^b \sigma(x)\,; \sigma \in \mathbf{S}([a,b]) \; and \; f \not< \sigma \right\}$$

($\mathbf{S}([a,b])$ *denoted the set of step functions on* $[a,b]$).

The transreal integral agrees with real integral. That is, if $a, b \in \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ is a bounded function then $_{\mathbb{R}^T}\int_a^b f(x) = \int_a^b f(x)$. Furthermore, if the improper integral is absolutely convergent then $_{\mathbb{R}^T}\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} f(x)$.

With the extension of the calculus to the transreal domain the physical applications which use derivatives and integrals can be considered at singularities.

3. Arithmetica Universalis

3.1 Individibles are independent from the Arithmetica Universalis

As a young man, Newton, writing in English, makes notes on the indivisibles. See (WHITESIDE, 1974, v. 1), from p. 89. The indivisibles were supplanted, long after Newton's death, by the modern development of infinitesimal numbers. Newton writes that surfaces are made up of infinitely many lines so that any surface is infinitely larger than a line but that one surface can be a different size from another so that infinity has a dual nature: infinities may be different sizes but they are all the same size when compared to a finite number. We shall not detain the reader with a discussion of the merits of this view but press on to consider the arithmetical treatment that Newton pursues on page 89, *ibid*:

> Thus $\frac{2}{0}$ is double to $\frac{1}{0}$ & $\frac{0}{1}$ is double to $\frac{0}{2}$, for multiply y^e 2 first & divide y^e 2^{ds} by 0, & there results $\frac{2}{1} : \frac{1}{1}$ & $\frac{1}{1} : \frac{1}{2}$. yet if $\frac{2}{0}$ & $\frac{1}{0}$ have respect to 1 they beare y^e

> same relation to it[,] y^t is $1{:}\frac{2}{0}{::}1{:}\frac{1}{0}$. & ought therefore to bee considered equall in respect of a unite.

That is Newton multiplies the first two terms, $2/0$ and $1/0$, by zero, $0/1$, and cancels the zeros, 0, to obtain $\frac{2}{0} \times \frac{0}{1} = \frac{2}{1}$ and $\frac{1}{0} \times \frac{0}{1} = \frac{1}{1}$. He then divides the remaining two terms $\frac{0}{1}$ and $\frac{0}{2}$ by zero, $\frac{0}{1}$, and cancels zeros, 0, to obtain $\frac{0}{1} \div \frac{0}{1} = \frac{0}{1} \times \frac{1}{0} = \frac{1}{1}$ and $\frac{0}{2} \div \frac{0}{1} = \frac{0}{2} \times \frac{1}{0} = \frac{1}{2}$. He then states, without proof, that $1{:}\frac{2}{0}{::}1{:}\frac{1}{0}$.

In particular, Newton holds that $\frac{0}{0} = 1$ and $\frac{0}{a} = \frac{0}{b} \Leftrightarrow a = b$. This is inconsistent with both real and transreal arithmetic but is coherent if we regard 0 as a fixed, infinitesimal number with a multiplicative inverse. This theme, of cancelling infinitesimals or derivatives, recurs throughout Newton's work but is absent from his presentation of the Arithmetica Universalis. We conclude that Newton could obtain the limit of some functions whose denominator *asymptotes* to zero, either by reasoning about indivisibles or using his fluxions, but could not divide by an *exact* zero. The same holds, *mutatis mutandis*, for real arithmetic and modern calculus.

3.2 Arithmetica Universalis Versus Real and Transreal Arithmetic

Newton's lecture notes on arithmetical algebra survive. See (WHITESIDE, 1974, v. 5) especially pages 61-107. These methods later became known as the Arithmetica Universalis. On a cursory reading, Newton's methods might appear, to the modern reader, to be just algorithms for implementing real arithmetic but this is not so - there are two differences.

The first difference is Newton's definition of equality. "Equations - the collection together of quantities which are equal to one another or effectively equal to nothing... $x = p$. OR $x - p = 0$." (WHITESIDE, 1974, v. 5, p. 111).

We read this as saying that $x = p$ is a synonym for $x - p = 0$, in other words, $a = b$ if and only if $a - b = 0$. Our reading is more limiting than the modern definition of mathematical equality which allows equivalent mathematical objects to be held equal, regardless of whether or not they are numbers subject to an additive identity. Newton's definition of equality is too tight, it blocks some developments of real arithmetic that the modern definition of equality allows; in particular it blocks transreal arithmetic, which is a superset of real arithmetic. Transreal arithmetic holds that $\infty = \infty$ and $\infty - \infty = \Phi \neq 0$. In other words, infinity is equal to itself but does not have an additive inverse - contrary to our reading of Newton's definition of equality.

The second difference is that the operations of the Arithmetica Universalis are not defined tightly enough to exclude mathematical structures that lie outside real arithmetic. In particular all of the transreal operations of addition, subtraction, multiplication, and division appear in Newton's corresponding operations. The reader may confirm this by comparing the tutorial in the Section Tutorial on Transreal Arithmetic and in (ANDERSON, 2011) with the operations in (WHITESIDE, 1974, v. 5), especially pages 61-107. This comparison is made, briefly, in the following subsection.

In short, Newton used an early form of arithmetic which, with suitable modifications, can be formalised, *inter alia*, as real arithmetic or as transreal arithmetic.

3.3 Similarities Between Transreal Arithmetic and the Arithmetica Universalis

In this subsection all page numbers refer to the translation of Newton in (WHITESIDE, 1974, v. 5); all equation numbers refer to our equations in Section Tutorial on Transreal Arithmetic.

For the most part Newton did not impose side conditions on his definitions; specifically he did not forbid division by zero in the Arithmetica Universalis so when we read his definitions literally they admit transreal arithmetic.

Page 61-62:

> A quantity set below a quantity with a short intervening rule denotes their quotient, that is, the quantity which arises from the division of the upper quantity by the lower one. ... $\frac{a}{b}$ denotes that which arises on dividing a by b - if, say, a be 15 and b be 3, then it denotes 5. ... Quantities of this sort, however, are called fractions, the upper part the numerator and the lower one the denominator.

This establishes a commonality between this one of Newton's notations and ours (Newton used several notations for the arithmetical operations, only some of which have survived into modern use).

Page 75: "Fractions are multiplied by 'drawing' numerators into numerators and denominators into denominators. Thus ... a/b times c/d makes ac/bd ..." This is given, in our tutorial, as equation (2.1). Note that Newton does not specify any side conditions on multiplication so a literal reading of his rule admits zero numerators and zero denominators; hence, on a literal reading, Newton's multiplication applies to all transreal numbers.

Page 81:

> When, indeed, those quantities are fractional, multiply the numerator of the quantity to be divided into the denominator of the divisor, and its denominator into the numerator: the former product will then be the numerator of the quotient, the latter one its denominator. Thus to divide $\mathrm{a/b}$ by $\mathrm{c/d}$, there is written $\mathrm{ad/bc}$, where, of course, a is multiplied by d and b by c.

Our equation (2.2) gives the same result:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$$

Thus a literal reading of Newton admits transreal division.

Page 67: "Positive fractions whose denominator is the same are united by adding the numerators. Thus $\frac{1}{5} + \frac{2}{5}$ make $\frac{3}{5}$; and $2ax/b\ +\ 3ax/b$ make $5ax/b$ ... Negative quantities are added in the same way as positive ones." This admits the transreal addition of infinities, equations (2.3) to (2.6). However, it would give a different result for adding a signed infinity to nullity. A literal reading of Newton would admit, for example, $\Phi + \infty = 0/0 + 1/0 = (0 + 1)/0 = 1/0 = \infty$ where transreal arithmetic has $\Phi + \infty = 0/0 + 1/0 = (0 \times 0 + 1 \times 0)/(0 \times 0) = 0/0 = \Phi$.

Page 107:

> This reduction is, however, of especial use in the addition and subtraction of fractions; for, should they have differing denominators, they must be reduced to the same one before they can be united. Thus $\mathrm{a/b} + \mathrm{c/d}$ comes by reduction to be $\mathrm{ad/bd} + \mathrm{bc/bd}$, that is, $(\mathrm{ad} + \mathrm{bc})/\mathrm{bd}$.

This is given by our equation (2.7) so a literal reading of Newton admits transreal addition of fractions without a common denominator.

Page 67: "Negative quantities are added in the same way as positive ones. ... $11ax/b$ and $-4ax/b$ make $7ax/b$ ..." This admits transreal subtraction, equation (2.8). In particular, in our notation, $11ax/b - 4ax/b = 11ax/b + (-4ax)/b$ has its right hand side rendered by Newton's: "$11ax/b$ and $-4ax/b$" which means $11ax/b + (-4ax)/b$. All computation paths producing the same result $7ax/b$.

Page 71: Our transreal subtraction is also admitted by Newton's "In algebraic terms subtraction is done by connecting quantities with all signs changed in the subtrahend and, moreover, uniting what can be united, exactly as happened in addition." Thus $11ax/b - (+4ax)/b$ has subtraction '$-$' rendered as addition '$+$' with the sign of the subtrahend $(+4ax)/b$ changed to $(-4ax)/b$ giving $11ax/b - 4ax/b = 11ax/b + (-4ax)/b$ exactly as in our equation (2.8).

Taking all of this together, a literal reading of Newton admits the whole of transreal arithmetic when we take care to add two infinities using Newton's algorithm for adding fractions with a common denominator and to add all other combinations of numbers using Newton's algorithm for adding fractions with different denominators.

4. Laws of Motion

4.1 Arithmetical Laws of Motion

We are now ready to cast Newton's laws of motion into transreal arithmetic. We work with a modern translation of Newton in (COHEN and WHITMAN, 1999, p. 416-7) and tackle the laws in order of increasing difficulty: Law 1, Law 3, Law 2.

Law 1: "Every body perseveres in its state of being at rest or of moving uniformly straight forward except in so far as it is compelled to change its state by forces impressed."

We read this as implying that a nett force must be non-zero and non-nullity to compel a change of state. In other words, a particle accelerates only in response to a positive or negative force.

Law 3: "To any action there is always an opposite and equal reaction; in other words, the action of two bodies upon each other are always equal and always opposite in direction."

We read this as implying that an active force, F, is opposed by a reactive force, $-F$, where the negation is computed using transreal arithmetic. As Newton explains, the assignment of the terms 'action' and 'reaction' is arbitrary.

Law 2: "A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."

We read this, in the usual arithmetical notation, as $F = ma$, computed with transreal arithmetic.

This completes the transarithmetical reading of Newton's laws. By virtue of the totality of transreal arithmetic (ANDERSON et al, 2007), Transreal Newtonian Physics allows division by zero, which immediately generalises Laws 1 and 3.

Let us examine the second law in more detail. Table 1 shows a round-trip computation of acceleration and mass using transreal arithmetic. We use this computation to establish commutativities. The first column of the table, headed m, shows the given mass of a Newtonian particle. This mass is a fixed, positive, real number, r_1, such that $0 < r_1 < \infty$. The second column, headed a, shows the given, non-negative, transreal acceleration of the particle. Within the column, r_2 is a fixed, positive, real number, such that $0 < r_2 < \infty$. The reader may easily extend the table to deal with negative accelerations. The third column, headed F, shows the computation of force as $F = ma$. The fourth column, headed a', shows the acceleration computed as $a' = F/m$. The last column, headed m', shows the mass, computed as $m' = F/a$. All computations are in transreal arithmetic.

m	a	F	a'	m'
r_1	0	0	0	Φ
r_1	r_2	$r_1 r_2$	r_2	r_1
r_1	∞	∞	∞	Φ
r_1	Φ	Φ	Φ	Φ

Table 1. Transrithmetical computation of Newton's second law

Recall that transreal division is a proper superset of multiplication by the multiplicative inverse. In general, transreal division does not introduce any kind of inverse but, in restricted cases, it may do so. The arithmetical restrictions inherent in Newtonian particles ensure that acceleration is completely specified by division, which is shown by the second (a) and fourth (a') columns having identical entries within the body of the table: $(0,0)$, $(r_2\,, r_2)$, (∞, ∞), (Φ, Φ). The mass is underspecified by division, as is shown by the first (m) and last (m') columns having three different entries in the body of the table: three occurrences of (r_1, Φ). This is to be expected: for example, any positive, finite mass that undergoes zero acceleration gives rise to a zero force - but we cannot then deduce the mass from knowing only that the force and acceleration are both zero! The entries of nullity in the last column may be interpreted as indicating that the computation path followed yields no information about the given (or 'true') value of the mass.

On a classical understanding of the universe this is of no consequence to how particles move though it does, of course, block the computation of mass in some cases. In other words the transreal motions provide for determined motions, of particles with a given mass, but do not always allow the computation of the mass from the observed motions of these primary particles. We may anticipate that other observations would allow the computation of their mass, such as measurements of the motion of secondary particles engaged in gravitational attractions with the primary ones. For example, if a mass the size of the Earth were unmoving then we could not calculate its mass from knowing that its acceleration is zero and the force acting on it is zero but, were a pair of perfect twins to visit the mass at antipodal points, they could still jump up and down on it in synchrony!

When using the second law, the computed values of force and acceleration are completely determined. We make this absolutely explicit by introducing two transarithmetical corollaries of Newton's verbal statement of his second law. The corollaries are computed in transreal arithmetic.

Law 2a: $F = ma$ when $0 < m < \infty$ and a is transreal.

Law 2b: $a = F/m$ when $0 < m < \infty$ and F is transreal.

This leaves the computation of the sometimes underspecified mass. We define:

Law 2c: $m = F/a$ when a, F are transreal. When $m \in \mathbb{R}$ the computed mass, m, is determined but when the computed mass $m = \Phi$ the true, finite mass is hidden.

Thus Law 2c computes either the true mass or else flags that the mass is hidden. We cannot ask much more of the arithmetic, that it computes the modelled values for particles in motion or else flags hidden variables in our model of physics; but we might do rather more by modelling a wider range of physical interactions using transnumbers as a new mathematical basis for our models. That is we might do more by developing a Transreal Newtonian Physics. We take some preliminary steps toward this in the Section Discussion.

4.2 Differential Laws of Motion

Laws 1 and 3 are already completely generalised by their transreal reading developed in the subsection immediately above. It remains only to re-write Laws 2a, 2b, 2c in differential form. We use Leibnizian notation because this is more widely used in Mathematical Physics. It is to be understood that derivatives are computed using the transreal definitions given in (REIS and ANDERSON, 2015). As usual v is the velocity and t is the time.

Law 2a: $F = m\frac{\mathrm{d}v}{\mathrm{d}t}$ when $0 < m < \infty$ and $\frac{\mathrm{d}v}{\mathrm{d}t}$ is transreal.

Law 2b: $\frac{\mathrm{d}v}{\mathrm{d}t} = F/m$ when $0 < m < \infty$ and F is transreal.

Law 2c: $m = F \div \frac{\mathrm{d}v}{\mathrm{d}t}$ when $\frac{\mathrm{d}v}{\mathrm{d}t}$ and F are transreal. When $m \in \mathbb{R}$ the computed mass, m, is determined but when the computed mass $m = \Phi$ the true, finite mass is hidden.

The reader is now free to generalise differential equations wherever they occur in Mathematical Physics or more widely.

5. Discussion

In this section we discuss how transmathematics may be used to model physical situations.

5.1 Seesaw

In this subsection we consider a very simple physical experiment and analyse it in both conventional and unconventional ways. The point being to illustrate the trials and reversals that arise when exploring a problem with transmathematics for the first time. The story we are about to recount may help readers in their own explorations of more important problems.

Suppose we have a seesaw, that is a bar balanced on a fulcrum. We place a weight of mass m_1 at a distance d_1 on one side of the fulcrum and place a weight of mass m_2 at a distance d_2 on the other side of the fulcrum. When is the seesaw balanced? The usual answer is when $m_1d_1 = m_2d_2$ and we call each m_id_i a 'moment.' But we may hypothesis another solution: the seesaw is balanced when $m_1/d_2 = m_2/d_1$. This gives the usual answer for m_i, d_i finite and non-zero. Suppose we take $m_1 = 1$ and $m_2 = 2$ with $d_1 = d_2 = 0$ then $m_1/d_2 = 1/0 = \infty = 2/0 = m_2/d_1$ and the seesaw is balanced. Suppose we take $m_1 = m_2 = d_1 = d_2 = 0$ then $m_1/d_2 = 0/0 = \Phi = 0/0 = m_2/d_1$ and the seesaw is balanced. All of these non-finite results agree with experiment. Is there any empirical reason to prefer one formula over the other?

Suppose we add a third weight of mass m_3 at distance $d_3 \neq d_2$ on the same side of the fulcrum as m_2. When is the seesaw balanced? The usual answer is when $m_1 d_1 = m_2 d_2 + m_3 d_3$. We are at a loss to find any simple formula involving division so Occam's Razor drives us to accept the usual solution, which we now do.

Suppose we have two infinite weights at the fulcrum. That is $m_1 = m_2 = \infty$ and $d_1 = d_2 = 0$ then $m_1 d_1 = \infty \times 0 = \Phi = \infty \times 0 = m_2 d_2$ and the seesaw is balanced.

Is there any non-finite case that violates our expectations? Consider two nullity weights $m_1, m_2 = \Phi$ and a finite weight $m_3 = 3$ at distances $d_1 = 1,\ d_2 = 2,\ d_3 = 3$. Our usual expectation is that the seesaw is balanced when $m_1 d_1 = m_2 d_2 + m_3 d_3$ that is when $\Phi \times 1 = \Phi \times 2 + 3 \times 3$ which simplifies to $\Phi = \Phi$ and we expect the seesaw to be balanced regardless of the position and mass of the weight m_3. But this violates our expectations because the nullity masses impart a nullity force and the transreal reading of Law 1 tells us that an object accelerates only in response to a positive or negative force, not a zero or nullity force. Therefore, contrary to our expectations, the formula tells us that the seesaw is balanced when it has a single, off centre, force acting on it.

In the absence of experiment, this violation of our expectations has an easy remedy. We sum only the positive moments. Alternatively we transform nullity moments into zero moments before summing moments. In either case we are modelling the physics of the seesaw not with a purely arithmetical formula but with a formula that has some conditionality to it. This additional complexity makes our solution total. We gain the ability to compute the moments on the seesaw even in the case of mathematical singularities involving non-finite numbers that have a zero denominator.

The seesaw is a simple, but non-trivial, example of computing Newtonian solutions at mathematical singularities. It involves easily realisable situations which give rise to non-finite solutions in some models of the experiment, solutions which agree with experimental evidence!

In the next subsection we use Transreal Newtonian Physics to make a prediction about black holes that is potentially capable of astronomical verification and verification in high-energy accelerators.

5.2 Black Hole

In this subsection we make predictions about violent physical singularities that are subject to empirical confirmation or rejection. In doing this we draw on Transreal Newtonian Physics but treat is as an analogy to make predictions about physical behaviour *exactly* at a real singularity in a black hole - a situation that neither conventional mathematics nor physics can presently analyse.

Suppose that two massive, Newtonian particles, with electrostatic charges of the same polarity, are drawn into the singularity inside a black hole. At the singularity the particles experience an infinite gravitational attraction and an infinite electrostatic repulsion. The nett force is $\infty - \infty = \Phi$. By our transreal reading of Law 1, the particles are free to move but experience no force compelling them to move. Let us suppose, which modern Physics teaches, that it is in the nature of particles to undergo stochastic motion. Now the particles may move stochastically away from the singularity, as a pair of particles moving in opposite directions, as required by Law 3. This creates a convection current, as massive, charged particles fall into a singularity and escape it stochastically, only to fall back in again unless the stochastic motion carries them beyond the event horizon with an escape velocity away from the black hole!

Of course we do not expect real black holes to operate according to any reading of Newtonian physics, even one extended with notions of an event horizon and the stochastic motion of particles, but we may generalise our argument. If, as we are taught by modern Physics, the universe escaped from a singularity at the big bang then there is some process for escaping singularities and we hypothesise that it is a process involving nullity force at the singularity and stochastic motion everywhere. At this level of analysis we are not concerned with the nature of the stuff that escapes the singularity; it may be space, time, energy, mass or anything at all. It is of no concern. We argue only that this process operates today in the singularities of a black hole and that this stuff, whatever it is, circulates in a convection current. Hence we predict that the event horizon of a black hole is not stable but rather roils in response to the shifting centre of mass brought about by the convection current. This prediction is potentially capable of astronomical verification; firstly by observing the x-ray emissions from gasses falling into a black hole which will be emitted at points that do not lie exactly on a radius about the black hole and with energies that are not monotonically related to the distance from the black hole. We also note that a roiling surface has a higher surface area than a smooth surface so a real black hole will experience more Hawking radiation than assumed in smooth models. If small black holes can be made in particle accelerators then this prediction might be tested by measuring the mass and lifetime of the microscopic black holes.

We expect that professional physicists can make more and better predictions of the behaviour of physical systems at singularities. We simply offer transreal calculus and a transphysics as first steps toward generalising all of the mathematical tools that physicists use.

5.3 Computers

In this subsection we anticipate future developments in the design of digital computers and programming languages. Almost all general-purpose computers use the von Neumann architecture but this makes a physically impossible assumption. It assumes that data move in unit time, regardless of the distance over which they travel. Thus the von Neumann architecture assumes that data violate the light speed constraint that appears to hold in our universe. The von Neumann architecture handles this by stalling the processor until data are ready but there is another approach known to Computer Science. Dataflow machines assume that data travel at a finite speed. Such machines have very high performance in special computations but it is difficult to design a useful, general-purpose, dataflow machine. One stumbling block is that ordinary computer instructions have logical exceptions which are handled, in dataflow machines, by re-routing data in ever more complicated patterns. However, it is possible to design total instruction sets that have no logical exceptions, leading to much more effective dataflow machines. Such instruction sets may profitably use transreal arithmetic in their arithmetical operations.

We may expect that total instruction sets will make it much easier for a compiler to compose subroutines automatically. Providing the parameters are of the correct type, functions can be composed arbitrarily. This should make it much easier to implement optimisation systems that have some component of stochastic or deterministic recombination of programmed components. It may also makes metaprogramming simpler so that computer systems can more easily generate efficient programs to deal with changed circumstances.

Total instruction sets also have an important role to play in safety critical systems. In a suitable architecture, any syntactically correct program is semantically correct - in the sense that it will not crash on a logical error. Thus any program that compiles, is guaranteed to execute without such errors. This is valuable in itself and makes it easier to verify programs.

Such developments in hardware and software might be realised in the future and, if they are, we may expect that programs will execute more quickly if they implement total systems rather than the partial systems ordinarily programmed today. The current paper is a first step toward expressing physical problems in transmathematics so that a wide range of scientific and engineering problems can be programmed efficiently on such machines. This gives a pragmatic reason to want a transmathematics and transphysics.

6. Conclusion

We go back to the primary sources of Newton's papers and show that his mathematical methods extend to the transreal numbers so that Transreal Newtonian Physics operates at mathematical singularities. We demonstrate Transreal Newtonian Physics with a simple, but non-trivial, example of non-finite forces applied to a beam, balanced on a fulcrum. We then analyse the singularity in a black hole using Transreal Newtonian physics and find that massive charged particles circulate through the singularity and its neighbourhood in a convection current. Arguing by analogy, we predict that the event horizon of real black holes is not stable but roils in response to perturbations of the centre of mass, brought about by convection current at the physical singularity inside the black hole. This leads us to propose experiments, in both astronomy and high energy physics, that might confirm or rebut our predictions. We expect that professional physicists can make more and better predictions than we can. We finish by noting some of the practical advantages that arise from total computation.

Referências Bibliográficas

ANDERSON, J. A. D. W. Perspex machine IX: Transreal analysis. *Vision Geometry XV Proceedings of the SPIE*, v. 6499, p. 1-12, 2007. Disponível em <http://bookofparagon.com/Mathematics/PerspexMachineIX.pdf > Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W. Perspex machine XI: Topology of the transreal numbers. In: INTERNATIONAL MULTICONFERENCE OF ENGINEERS AND COMPUTER SCIENTISTS, 2008. Hong Kong. Proceedings... International Association of Engineers, 2008. p. 330-338. Disponível em <http://bookofparagon.com/Mathematics/PerspexMachineXI.pdf> Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W. Evolutionary and revolutionary effects of transcomputation. In 2nd IMA Conference on Mathematics in Defence. Proceedings... Institute of Mathematics and its Applications, 2011. Disponível em <http://bookofparagon.com/Mathematics/Evolutionary_Revolutionary_2011_web.pdf> Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W. Trans-floating-point arithmetic removes nine quadrillion redundancies from 64-bit IEEE 754 floating-point arithmetic. In: INTERNATIONAL CONFERENCE ON COMPUTER SCIENCE AND APPLICATIONS, 2014. San Francisco. Proceedings... International Association of Engineers, 2014, p. 80-85. Disponível em <http://www.iaeng.org/publication/WCECS2014/WCECS2014_pp80-85.pdf> Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W.; GOMIDE, W. Transreal arithmetic as a consistent basis for paraconsistent logics. In: INTERNATIONAL CONFERENCE ON COMPUTER SCIENCE AND APPLICATIONS, 2014. San Francisco. Proceedings... International Association of Engineers, 2014. p. 103-108. Disponível em <http://www.iaeng.org/publication/WCECS2014/WCECS2014_pp103-108.pdf> Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W.; REIS, T. S. dos. Transreal limits expose category errors in IEEE 754 floating-point arithmetic and in mathematics. In: INTERNATIONAL CONFERENCE ON COMPUTER SCIENCE AND APPLICATIONS, 2014. San Francisco. Proceedings... International Association of Engineers, 2014. p. 86-91. Disponível em <http://www.iaeng.org/publication/WCECS2014/WCECS2014_pp86-91.pdf> Acesso em 02 de junho de 2015.

ANDERSON, J. A. D. W.; VÖLKER, N.; ADAMS A. A. Perspex Machine VIII: Axioms of transreal arithmetic. *Vision Geometry XV Proceedings of the SPIE*. v. 6499, p. 649903.1-649903.12, 2007. Disponível em <http://bookofparagon.com/Mathematics/PerspexMachineVIII.pdf> Acesso em 02 de junho de 2015.

COHEN, I. B.; WHITMAN, A. *Isaac Newton The Principia, Mathematical Principles of Natural Philosophy*. University of California Press, 1999.

GOMIDE, W; REIS, T. S. dos. Números Transreais: Sobre a Noção de Distância. *Synesis*, v. 5, n. 2, p. 197-210, 2013. Disponível em <http://seer.ucp.br/seer/index.php?journal=synesis&page=article&op=view&path[]=413&path[]=241> Acesso em 02 de junho de 2015.

NEWTON, I. Philosophiae Naturalis Principia Mathematica. Coloniae Allobrogum, Sumptibus Cl. & Ant. Philibert Bibliop. & Typogr., 1760.

REIS, T. S. dos. Números transreais: matemática ou devaneio? In: 14º SEMINÁRIO NACIONAL DE HISTÓRIA DA CIÊNCIA E DA TECNOLOGIA, 2014. Belo Horizonte. Proceedings... Sociedade Brasileira de História da Ciência, 2014. Disponível em <www.14snhct.sbhc.org.br/arquivo/download?ID_ARQUIVO=1899> Acesso em 02 de junho de 2015.

REIS, T. S. dos; ANDERSON, J. A. D. W. Transdifferential and transintegral calculus. In: INTERNATIONAL CONFERENCE ON COMPUTER SCIENCE AND APPLICATIONS, 2014a. San Francisco. Proceedings... International Association of Engineers, 2014. p. 92-96. Disponível em <http://www.iaeng.org/publication/WCECS2014/WCECS2014_pp92-96.pdf> Acesso em 02 de junho de 2015.

REIS, T. S. dos; ANDERSON, J. A. D. W. Construction of the transcomplex numbers from the complex numbers. In: INTERNATIONAL CONFERENCE ON COMPUTER SCIENCE AND APPLICATIONS, 2014b. San Francisco. Proceedings... International Association of Engineers, 2014. p. 97-102. Disponível em <http://www.iaeng.org/publication/WCECS2014/WCECS2014_pp97-102.pdf> Acesso em 02 de junho de 2015.

REIS, T. S. dos; ANDERSON, J. A. D. W. Transreal Calculus. *IAENG International Journal of Applied Mathematics*, v. 45, n. 1, p. 51-63, 2015. Disponível em <http://www.iaeng.org/IJAM/issues_v45/issue_1/IJAM_45_1_06.pdf> Acesso em 02 de junho de 2015.

REIS, T. S. dos; GOMIDE, W; KUBRUSLY, R. Números transreais: mais uma etapa na história dos números. In: SCIENTIARUM HISTÓRIA: VI CONGRESSO DE HISTÓRIA DAS CIÊNCIAS, DAS TÉCNICAS E EPISTEMOLOGIA, 2013. Rio de Janeiro. Proceedings... Universidade Federal do Rio de Janeiro, 2013. Disponível em <http://www.hcte.ufrj.br/downloads/sh/sh6/SHVI/trabalhos%20orais%20completos/trabalho_081.pdf> Acesso em 02 de junho de 2015.

WHITESIDE, D. T. *The Mathematical Papers of Isaac Newton* (8 volumes). Cambridge University Press, 1974.

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ANDERSON, J. A. D. W.; REIS, T. S. dos. Transreal Newtonian Physics Operates at Singularities. **Synesis**, v. 7, n. 2, dez. 2015. ISSN 1984-6754. Disponível em: http://seer.ucp.br/seer/index.php?journal=synesis&page=article&op=view&path%5B%5D=738. Acesso em: 30 Dez. 2015.