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MAPPING OF RADIANT REGIONS IN VIBRATING RECTANGULAR PLATES WITH DIFFERENT BOUNDARY CONDITIONS THROUGH THE SUPERSONIC AND USEFUL INTENSITIES

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Abstract. This work investigates the acoustical intensity generated by the sound radiated from rectangular plates in four cases with distinct combinations of classical boundary conditions. The objective is to identify the regions of these plates that effectively contribute to the radiated sound power into the far-field, especially when the driven frequency occurs below the critical coincidence frequency. This identification is done by filtering the non-propagating waves, both using the (analytical) supersonic intensity method and the (numerical) useful intensity model. Brief theoretical formulations, both for the plates vibration and the resulting acoustical field, are discussed. The closed form solution of the normal velocity field for the four cases is given. Then, the supersonic intensity is estimated. In the numerical examples, the comparison of the supersonic intensity, the useful intensity and the classical acoustic intensity is shown.

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1 INTRODUCTION

Rectangular plates are structural elements of great importance, being present in several places and in many engineering applications, such as on mechanical and aeronautic ones. In this way, the discussion and analysis of the sound radiation due to plates vibration has received the special attention of researchers in the last decades [2, 4, 9, 13].

A relevant concept was introduced by [13], named by the author as *supersonic intensity* (SI), to identify the regions on the source surface that radiates efficiently and contributes to the sound power. The calculus of the SI is originally based on the spatial Fourier transform, being formulated in the wavenumber space, where the sound field is processed. Its fundamental principle relies on the filtering of the non-propagating waves — the evanescent ones [14]. As a consequence, only the propagating components remains. Williams [14] also concerns in distinguishing the SI from the classical acoustic intensity (AI), showing that in the near field there is energy recirculation and the AI is not as reliable as the SI to identify the actual sound sources. However, his work is focused in rectangular plates simply supported in the four edges. He shows that it is possible to identify the corner, border and surface modes, in conformity with the results found in the literature [10].

Grande *e*t al. [4] examined the concept of supersonic intensity and reformulate it as a filtering operation directly in the space domain. The method allows the identification of the efficient radiation regions of a sound source and evaluates the actual contribution to the radiated sound power. A numerical example of a plate clamped in all edges is presented as well as an experimental study to illustrate the method and to examine its advantages and limitations. The method has the convenience of not requiring to move to the wavenumber space, but it is still restricted to separable geometries.

Corrêa and Tenenbaum [2] present an innovative technique for the computation of a numerical equivalent to the SI, for sound sources with arbitrary geometry, which was named by the authors as *useful intensity* (UI). As the SI, the technique filters the non-propagating components, extracting the propagating ones. The method is entirely formulated in the vibrating surface. The sound power is obtained through a matricial operator that is related to the normal velocity field, by using the boundary element method (BEM). This operator is Hermitian, guaranteeing that its eigenvalues are real and the associated eigenvectors form a basis for the velocity distribution. The use of a stop criteria allows the removing of the non-propagating components. The great advantage of the method is to be applied to non-separable geometries.

Although there is a substantial number of works in the literature dealing with rectangular plates radiation [5, 8, 17], the analytical determination of the *hot spots* of such plates with other boundary conditions was not examined yet. In this work it is computed the supersonic intensity for rectangular plates with four distinct boundary conditions. Finally, the SI is compared with the classical acoustic and useful intensities.

The four considered cases in this study are S-S-C-S, C-C-C-C, S-C-C-C, and S-S-C-F. The letters indicate the boundary conditions at the edges $x = 0$, $x = a$, $y = 0$ and $y = b$, as seen in Fig. 1 and refers to *free* (F), *clamped* (C) and *simply supported* (S), as usual.

2 MATHEMATICAL BACKGROUND

2.1 Equations of motion for a thin plate vibration

Let us consider a rectangular plate in plane xy, with length a, width b and thickness h. The classical Kirchhoff theory for thin plates will be considered. In the frequency domain, the differential equation that governs the transverse displacements $w(x, y)$, independently of the

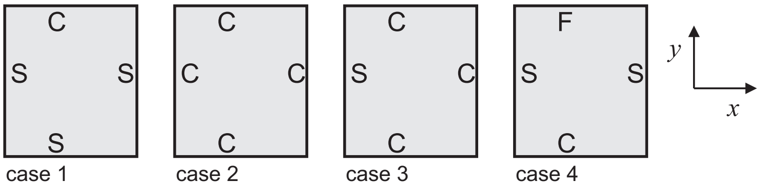

Figure 1: Four combinations of classical boundary conditions to be considered

imposed boundary conditions, is

$$\nabla^4 w(x,y) + k_b^4 w(x,y) = 0, \qquad \text{with} \qquad k_b^4 = \frac{\rho_s h \omega^2}{D}, \tag{1}$$

where k_b is the flexural wavenumber, ω is the circular frequency, and D is the flexural stiffness of the plate, given by

$$D = \frac{Eh^3}{12\rho_s(1-\nu^2)}, \tag{2}$$

where E is the Young modulus, ρ_s is the plate density and ν is the Poisson coefficient.

The normal velocity distribution $\hat{v}(x,y)$, in the frequency domain, is

$$\hat{v}(x,y) = i\omega w_{mn}(x,y), \tag{3}$$

where i is the imaginary unit, $w_{mn}(x,y)$ is the modal shape function associated with (m,n) mode, m and n being the mode counters, and ω_{mn} is the natural frequency for a given (m,n) mode. The function $w_{mn}(x,y)$ can be written as a product of beam functions, in the form [7]

$$w_{mn}(x,y) = \psi_m(x)\phi_n(y), \tag{4}$$

so that $\phi_m(x)$ and $\psi_n(y)$ are chosen as fundamental modal shapes for beams with the same boundary conditions of the corresponding plate. The functions

$$\psi_m(x) = C_1 \sin(\beta_1 x) + C_2 \cos(\beta_1 x) + C_3 \sinh(\beta_2 x) + C_4 \cosh(\beta_2 x); \tag{5}$$

$$\phi_n(y) = B_1 \sin(\mu_1 y) + B_2 \cos(\mu_2 y) + B_3 \sinh(\mu_2 y) + B_4 \cosh(\mu_2 y), \tag{6}$$

with a convenient choice of constants C_i and B_i, $i = 1, \ldots, 4$, that must satisfy the boundary conditions listed in Table 1, and $\beta_{1,2}$ and $\mu_{1,2}$ are, respectively, eigenvalues in the coordinates directions x and y.

Table 1: Boundary conditions for a beam

Notation	Type of support	Boundary conditions
a) S	Supported	$w = \frac{\partial^2 w}{\partial x^2} = 0$
b) C	Clamped	$w = \frac{\partial w}{\partial x} = 0$
c) F	Free	$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0$

It is worth noting that a closed form of the normal mode w_{mn} is obtained, for each case, by using the symplectic dual method, shown in [15]. Those authors use the variable separation method to solve the Hamiltonian dual form of the eigenvalue problem in thin vibrating plates (Eq. (1)).

For the other two opposite edges $x = 0$ e $x = a$, a characteristic equation and a set of eigenfunctions $\psi_m(x)$ can be obtained similarly.

The natural mode of vibration, given in Eq. (4) is obtained from the corresponding functions $\psi_m(x)$ and $\phi_n(y)$.

2.2 Analytical formulation of the supersonic acoustic intensity

The mathematical formulation of the radiation problem is widely discussed in [2]. However, for the sake of completeness and better understanding, the mathematical development is very briefly outlined here.

The supersonic acoustic intensity is a tool obtained using the inverse spatial Fourier transform to eliminate non-propagating (subsonic) waves, leaving the far-field radiating (supersonic) components.

The supersonic acoustic pressure, $\hat{p}^{(s)}$, and the normal supersonic velocity, $\hat{v}^{(s)}$, both in the frequency domain, that are associated with the corresponding pressure p and normal velocity v inside the radiation circle C_r, are written as

$$\hat{p}^{(s)}(x, y, 0, \omega) = \frac{1}{4\pi^2} \int\int_{C_r} \tilde{p}(k_x, k_y, 0, \omega) e^{i(k_x x + k_y y)} \, dk_x \, dk_y, \tag{7}$$

$$\hat{v}^{(s)}(x, y, 0, \omega) = \frac{1}{4\pi^2} \int\int_{C_r} \tilde{v}(k_x, k_y, 0, \omega) e^{i(k_x x + k_y y)} \, dk_x \, dk_y, \tag{8}$$

where $\tilde{p}$ and $\tilde{v}$ are, respectively, the pressure and normal component of velocity in the wavenumber domain, k_x and k_y are the wavenumbers in the plane directions, ω is the angular frequency and i is the imaginary unit, as usual [14].

The flow of acoustic energy that is radiated effectively into the far-field, i.e., the supersonic (acoustic) intensity (SI), is then defined as

$$\hat{I}^{(s)} = \frac{1}{2} \Re[\hat{p}^{(s)}(x, y, 0, \omega) \hat{v}^{(s)}(x, y, 0, \omega)^*], \tag{9}$$

where the subscript "*" denotes the complex conjugate and $\Re$ stands for real part.

Note that the supersonic intensity is essentially a spatial low-pass filtered version of the conventional active intensity, where the non-propagating waves are filtered out.

An important result shown by [13] is that the sound power, Π, calculated with the use of acoustic intensity (AI) is the same as that calculated with the SI. In other words, if S is the vibrating surface, then

$$\Pi = \int_S \hat{I}(x, y, 0, \omega) \, dS = \int_S \hat{I}^{(s)}(x, y, 0, \omega) \, dS, \tag{10}$$

3 NUMERICAL FORMULATION

It is well known that there is no closed form solution to compute the supersonic intensity for vibrating sound sources with arbitrary geometries. The alternative, in these cases, is to discretize the geometry and use numerical methods to model the radiation problem. These methods must be capable to identify, through computational tools, the regions of the source surface that effectively contributes to the radiated sound power. To obtain such numerical model, firstly the normal velocity field obtained in a closed form as presented in Subsection 2.1. The acoustical pressure field is then obtained with the *boundary element method* (BEM). In this work, the useful intensity according to the definition given by [2] is also shown for each of the four cases.

3.1 Formulation of the boundary element method

The boundary element method is based on Green's theorem to compute the fundamental solution of the Helmholtz equation to obtain an integral contour of the domain, called Kirchhoff-Helmholtz integral, given by

$$c\hat{p}(X,\omega) = \int_{\Gamma} \left(i\omega\rho_0\hat{v}(X_s,\omega)G(X_s|X) - \hat{p}(X_s,\omega)\frac{\partial G(X_s|X)}{\partial n_s} \right) d\Gamma, \tag{11}$$

where c is a coefficient dependent on the position of point $X(x,y,z)$; $X_s(x,y,0)$ is a point on the surface; Γ is the surface contour and $G(X_s|X)$ is the free field Green function, which is given by, see [12],

$$G(X_s|X) = \frac{e^{ik|X_s-X|}}{4\pi|X_s-X|}. \tag{12}$$

A matrix relationship between pressure and normal velocity at the surface can then be obtained as

$$\hat{p} = \mathbf{R}\hat{v} \tag{13}$$

Knowing $\hat{v}$ and $\hat{p}$, then it can be obtained the normal component of the acoustic intensity (AI), as

$$\hat{I} = \frac{1}{2}\Re[\hat{p}\,\hat{v}^*]. \tag{14}$$

where $\mathbf{R} = H^{-1}G$ is called *surface operator*. For details about H and G matrices, see [6].

3.2 Useful intensity

Xu and Huang [16] showed that the sound power can be expressed in terms of the normal velocity distribution. Such relationship is given as

$$\Pi = \hat{v}^H\bar{\mathbf{Q}}\hat{v}, \tag{15}$$

where H indicates the conjugate transpose and $\bar{\mathbf{Q}}$ is a Hermitian operator, i.e., $\bar{\mathbf{Q}} = \bar{\mathbf{Q}}^H$, called *power operator* by [2]. The characteristic of being Hermitian is a fundamental one, since it guarantees that the eigenvalues are all real and that any base of the invariant sub-spaces form an orthonormal set for the velocity distribution of the surface [3].

To formulate the model for the calculus of the sound power filtering the non radiating modes it is convenient to decompose the power operator in the form

$$\bar{\mathbf{Q}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}, \tag{16}$$

where $\mathbf{V}$ is a matrix such that its columns are the eigenvalues of $\bar{\mathbf{Q}}$, $\mathbf{V}^{-1}$ is the inverse matrix of $\mathbf{V}$, and $\mathbf{D}$ is the diagonal matrix containing the eigenvalues of $\bar{\mathbf{Q}}$. Since the matrix $\mathbf{V}$ is unitary, one can write Eq. (16) as

$$\bar{\mathbf{Q}} = \mathbf{V}\mathbf{D}\mathbf{V}^H, \tag{17}$$

and, then, the sound power can be computed as

$$\Pi = \hat{v}^H \mathbf{V}\mathbf{D}\mathbf{V}^H \hat{v} = \sum_{i=1}^{r} \lambda_i \langle \hat{v}^H, \mathbf{V}_i \rangle \langle \mathbf{V}_i^H, \hat{v} \rangle \tag{18}$$

where r is the global number of eigenvalues λ_i of $\bar{\mathbf{Q}}$. Such eigenvalues are called eigenvalues of velocity, in analogy to the denomination given by [1] to the singular values. The $\mathbf{V}_i^H$ are the lines of $\mathbf{V}^H$ and $\mathbf{V}_i$ are the columns of $\mathbf{V}$. In other words, the eigenvectors of $\bar{\mathbf{Q}}$, are called own pattern of velocity, since they form a set of modes for the normal velocity distribution.

The series presented in Eq. (18) can be truncated for a limited number or radiation modes that effectively contribute to the sound power radiated. In this way, the modes associated to eigenvalues with negligible magnitude are discarded, obtaining an accurate approximation for the sound power. This means that

$$\Pi = \sum_{i=1}^{r} \lambda_i \langle \hat{v}^H, V_i \rangle \langle V_i^H, \hat{v} \rangle \approx \sum_{i=1}^{r_c} \lambda_i \langle \hat{v}^H, V_i \rangle \langle V_i^H, \hat{v} \rangle, \tag{19}$$

with $r_c < r$, where r_c is a sufficient quantity of retained modes.

Aiming at obtaining such a truncation procedure, the eigenvalues are organized — without loss of generality — in a decreasing order of absolute values, that means, $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_r|$.

To chose the best value $r_c < r$ of the series shown in Eq. (18), it is adopted a truncation criteria based on the similarity of the matrices D e Q and on the definition of the convergent series [11]. As a consequence, the following criteria is adopted

$$\left| \frac{t - \sum_{i=1}^{r_c} \lambda_i}{t} \right| \left| \frac{\Pi - \sum_{i=1}^{r_c} \lambda_i \langle \hat{v}^H, V_i \rangle \langle V_i^H, \hat{v} \rangle}{\Pi} \right| < \delta \tag{20}$$

where t is the trace of matrix $\bar{Q}$ and δ is a tolerance value prescribed.

The determination of the useful intensity is done in three steps. In the first one, the useful normal velocity $\hat{v}^{(u)}$ is obtained. For that, the orthonormality of the eigenvectors is used. Then, the useful velocity is written as a sum of the retained modes in Eq. (19), that means,

$$\hat{v}^{(u)} = \sum_{i=1}^{r_c} \langle \mathbf{V_i}^H, \hat{v} \rangle \mathbf{V}_i. \tag{21}$$

In the second step, the useful pressure, $\hat{p}^{(u)}$, is obtained from the insertion of $\hat{v}^{(u)}$ in Eq. (13), resulting in

$$\hat{p}^{(u)} = \mathbf{R}\hat{v}^{(u)}. \tag{22}$$

Knowing the useful normal velocity and the useful acoustical pressure one can go to the last step, the useful intensity computation, given by

$$\hat{I}^{(u)} = \frac{1}{2}\Re[\hat{p}^{(u)}\hat{v}^{(u)*}]. \tag{23}$$

4 NUMERICAL RESULTS

This section presents the results for the supersonic intensity (SI), as presented in Section 2.3, and the useful intensity (UI), as presented in Section 3.2, for rectangular plates with the four considered boundary conditions shown in Fig. 1. In the first subsection the results for a plate with all edges clamped, Case 2, are compared with those obtained by [4]. In the second subsection the results obtained for the other three cases are presented. For the sake of comparison, the classical acoustic and useful intensities (AI) are also computed and plotted for all cases.

4.1 Clamped plate

As aforementioned, in this section, the SI and the AI are compared to a steel rectangular plate of dimensions 0.5 m × 0.7 m, and 0.001 m thick with four edges clamped. The plate was driven in mode (4,10), as shown in Fig. 2, at the frequency of 950 Hz, below the coincidence frequency. It is worth noting that, according with the thin plate theory that classifies the modes in a corner or edge modes for vibrating plates driven at a frequency below the coincidence frequency [10], there is radiation efficiency only along to the boundaries, due the energy cancelling effect.

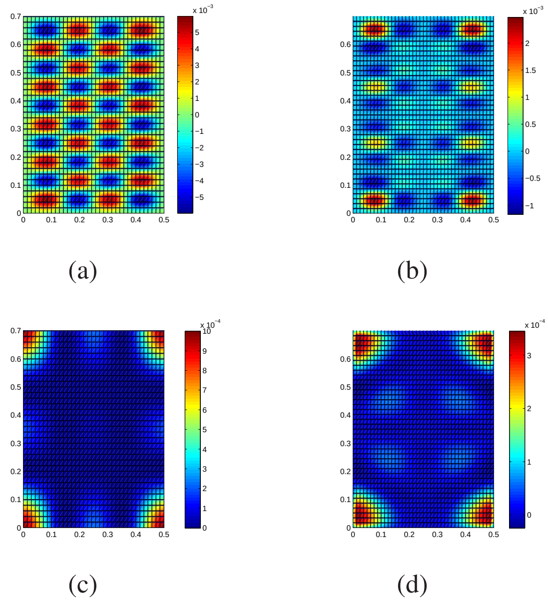

Figure 2: Boundary condition: Full clamped. (a) Normal velocity (m/s). (b) Acoustic intensity (W/m^2). (c) Supersonic intensity (W/m^2). (d) Useful intensity (W/m^2)

4.2 Other boundary conditions

This subsection presents the numerical results for the other three combinations of boundary conditions considered in this work for an aluminum rectangular plate of dimensions 1 m

$\times$ 1.2 m, and 0.002 m thick. In the sequel it is presented, for the other three boundary conditions, the considered mode with its corresponding frequency. The modes were selected to give a full panorama of corner and edge modes in according to the theory of radiation modes of [10]. Table 2 presents the excited modes for each boundary condition and the corresponding frequencies.

Table 2: Excited modes and corresponding frequencies for the considered boundary conditions

Boundary conditions	Excited mode	Frequency (Hz)
SSSC	(5,9)	388
SCCC	(5,7)	343
SSCF	(5,8)	372

In what follows, it is presented the distribution of four main fields, for each boundary condition. The first one (Subfigure a) is the *normal velocity field*, obtained in a closed form as presented in Subsection 2.1. The second one (Subfigure b) is the *classical acoustic intensity*, obtained also in closed form, by Eq. (14). The third one (Subfigure c) is the *supersonic intensity*, calculated as explained in Subsection 2.2. Finally, the fourth one (Subfigure d) shows the *useful intensity*, numerically computed as discussed in Section 3.2.

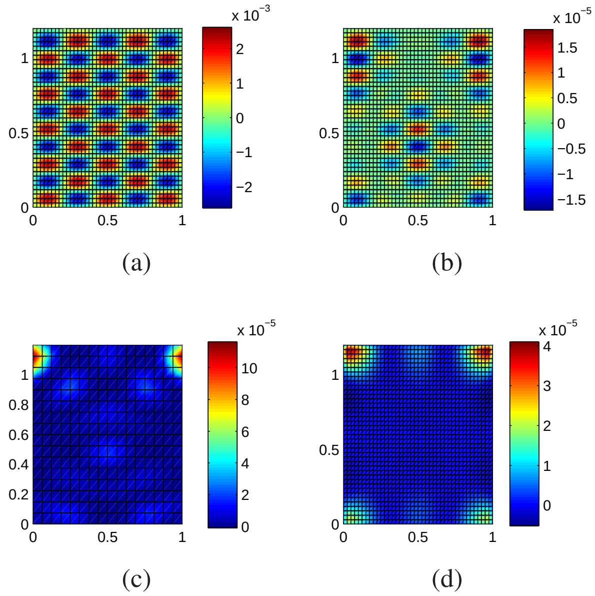

Figure 3: Boundary conditions S-S-S-C: (a) Normal velocity (m/s). (b) Acoustic intensity (W/m^2). (c) Supersonic intensity (W/m^2). (d) Useful intensity (W/m^2)

5 CONCLUSIONS AND REMARKS

In this research the acoustic intensity produced by the sound radiation from rectangular thin plates with four different combinations of boundary conditions is addressed. The analytical closed form to obtain the normal velocity field to each considered case was cast by the variables separation method to solve the Hamiltonian dual form [15].

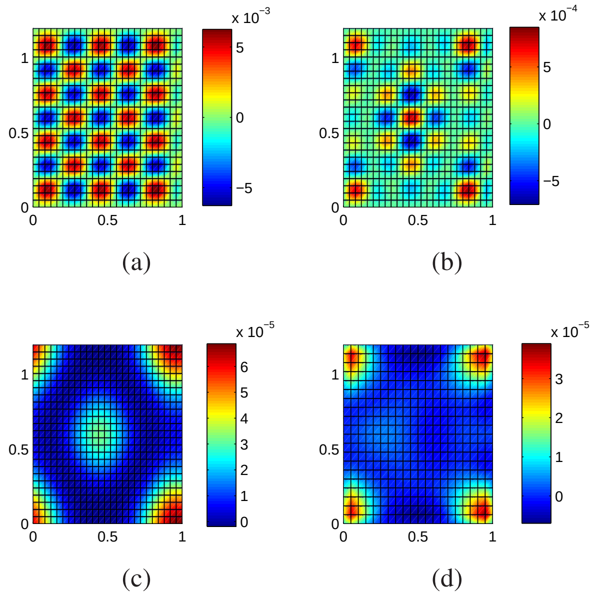

Figure 4: Boundary conditions S-C-C-C: (a) Normal velocity (m/s). (b) Acoustic intensity (W/m^2). (c) Supersonic intensity (W/m^2). (d) Useful intensity (W/m^2)

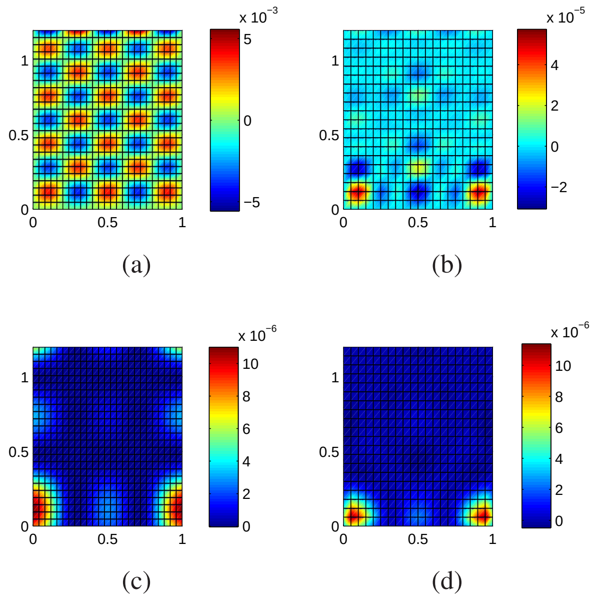

Figure 5: Boundary conditions S-S-C-F: (a) Normal velocity (m/s); (b) Acoustic intensity (W/m^2); (c) Supersonic intensity (W/m^2). (d) Useful intensity (W/m^2)

In the sequel, the mathematical formulation to obtain the supersonic intensity as suggested by [13] via spatial Fourier transform is briefly presented. The framework of the useful intensity, firmly based on the boundary element method is also shortly discussed. Then, the supersonic and the useful intensities for the distinct boundary conditions are computed. The most important features in these calculi are the identification of the regions, in each case, that effectively contribute to the radiated sound power. Although the useful intensity technique has been developed to assess the hot spots for non-separable geometries, it was applied here to rectangular plates, for validation purposes.

Four boundary conditions cases — two of them not found in the literature — were addressed. These cases are S-C-S-S, C-C-C-C, S-C-C-C and S-S-C-F. Three acoustic intensities were compared for all cases: The classical (also called active) acoustic intensity; the supersonic intensity, calculated analytically; and the useful intensity, computed numerically via the boundary element method and spectral decomposition [1]. The primary application of this research is, naturally, the noise control of vibrating plates.

The studied cases constitute a general panorama of the radiation modes, especially the corner and edge modes. Covering almost all boundary conditions for rectangular plates found in more complex structures, the presented results furnish a reliable source of data for researchers and designers to localize the regions of a structure that effectively radiate sound to the far-field.

REFERENCES

[1] G. V. Borgiotti. The power radiated by a vibrating body in an acoustic fluid and its determination from boundary measurements. *Journal of the Acoustical Society of America*, 88:1884–1893, 1990.

[2] C. A. Corrêa Jr. and R. A. Tenenbaum. Useful intensity: A technique to identify radiating regions on arbitrarily shaped surfaces. *Journal of Sound and Vibration*, 332:1567–1584, 2013.

[3] G. H. Golub and C. F. van Loan *Matrix Computation*. Johns Hopkins University Press, 1996.

[4] E. Fernandez-Grande, F. Jacobsen and Q. Leclere. Direct formulation of the supersonic acoustic intensity in space domain. *Journal of the Acoustical Society of America*, 12:186–193, 1996.

[5] S. H. Hosseini-Hashemi, K. Khorshidi and H. R. Taher. Exact acoustical analysis of vibrating rectangular plates with two opposite edges simply supported via Mindlin plate theory. *Journal of Sound and Vibration*, 332:883–900, 2009.

[6] F. Holmstrom. *Structure-acoustic analysis using BEM/FEM; implementation in MATLAB*. Master thesis, Lund University, Lund, Sweden, 2009.

[7] A. W. Leissa. *Vibration of plates.* NASA SP-169, Office of Technology Utilization, NASA, Washington, DC, 1969. Reprinted by The Acoustical Society of America, 1993.

[8] Y. Liu, W. Jiang and H. Wu. Analyzing acoustic radiation modes of blaffled plates with a fast multipole boundary element method. *Journal of Vibration and acoustics*, 135(1):011007–011007-7, 2013.

[9] M. B. S. Magalhães and R. A. Tenenbaum. Supersonic acoustic intensity for arbitrarily shaped sources. *Acta Acustica united with Acustica*, 92:189–201, 2006.

[10] G. Maidanik. Vibrational and radiative classifications of modes of a baffled finite panel. *Journal of Sound and Vibration*, 34:447–455, 1974.

[11] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Education, 3rd Ed., 1976.

[12] L. Stakgold. Green's Functions and Boundary Value Problems. *Wiley-Interscience Publications*, 1979.

[13] E. G. Williams. Supersonic Acoustic Intensity. *Journal of the Acoustical Society of America*, 97:121–127, 1995.

[14] E. G. Williams. Supersonic Acoustic Intensity on Planar Sources. *Journal of the Acoustical Society of America*, 104:2845–2850, 1998.

[15] Y. Xing and B. Liu. New exact solutions for free vibrations of rectangular thin plates by sympletic dual method. *Acta Mechanica Sinica*, 25(5):265–270, 2009.

[16] Z. Xu and Q. Huang. The study of three-dimensional structural acoustic radiation using FEM and BEM. *Advances in Theoretical and Applied Mechanics*, 4:189-194, 2010.

[17] J. W. Yoo. study on the general characteristics of sound radiation of a rectangular plate with different boundary edge conditions. *Journal of Mechanical and Technology*, 24(5):1111–1118, 2010.