REUCP, Petrópolis, Volume 9, n° 1 (2015), P. 31-41
ISSN 2318-0692

A MULTISCALE FINITE DIFFERENCE METHOD FOR LINEAR ELLIPTIC PROBLEMS IN HETEROGENEOUS MEDIA

Alexandre S. Francisco, Jorge J. Trujillo, and Leonardo M. Sacramento

Universidade Federal Fluminense, 27255-125, Volta Redonda, RJ, Brasil

Keywords: Multiscale method, Finite differences, Porous media.

Abstract. We present a multiscale finite difference method to solve problems of highly heterogeneous porous medium flow. Heterogeneity in permeability fields is assumed to occur on a wide range of length scales. Our multiscale method allows for incorporating fine-scale information of the permeability into coarse-scale iteration procedures. In this direction, we define multiscale basis functions in order to obtain discrete solutions in local problems. In another paper, such basis functions have already been defined by using hybridized mixed finite elements. In this paper, we use finite differences to do so. Our multiscale method is expected to be an inexpensive alternative to solve the global problem at only fine scale. Numerical results are presented to check the performance of the multiscale method.

E-mail addresses: afrancisco@metal.eeimvr.uff.br, jjuarezt@hotmail.com, leonardo_sacra@hotmail.com

1 INTRODUCTION

Fluid flow problems in porous media are governed by mass and *momentum* balance equations. The *momentum* equation for modelling porous medium flows is the Darcy law. This equation poses that the flow velocity in porous media is proportional to the pressure gradient. The permeability is the coefficient of this equation, which quantifies the capacity of the porous medium to displace fluids.

The permeability coefficient reflects the heterogeneity in porous media at all scales, which occurs from the porous scale to the scale of kilometers, for example. For computing accurately the velocity and pressure fields, it is necessary to account for the influence of fine-scale variations in the permeability coefficient to generate reliable solutions.

Combining the Darcy law and the mass equation, a second order linear elliptic equation is obtained. The development of multiscale methods for solving second order elliptic partial differential equations, related to liquids flowing in porous media as oil or underground water, has taken attention of several research groups worldwide [16]. Multiscale methods can provide large computational savings when applied as direct solvers for problems occurring on a broad range of heterogeneity scales.

In the evolution of the numerical methods used to reach an approximate solution of problems modelled by Laplace and Poisson equations in heterogeneous porous media, it is mandatory to refer to [2]. This work proposed in the decade of 1970 the concept of homogenization, which could be considered as a pioneering procedure for the solution of such kind of problems.

Other works regarding parallel programming and the adjustments needed in the numerical methods can be found in the decade of 1980, in order to make them able to take advantage of newer computational resources. Adjustments for parallelization consist of solving large domain problems dividing them by parts or blocks, so they can be iterated separate and simultaneously in different processors, or clusters, in order to reduce the execution time.

An iterative procedure that was made in order to be adaptable for massively parallel programming has used a development based on the technique of domain decomposition and the association of each subdomain to a specific processor [6]; it is one of the first references regarding the development of such procedures.

The *MFEM* (Multiscale Finite Element Method) [11] was considered fully parallelizable and naturally adaptable for massively parallel computers to solve porous medium problems. It captures the effects of the small scales and reflects them on larger scales without solving all details at minor scales; and it introduces a scheme with base functions. Later on, their authors have presented an evolution of the same method for elliptic problems with fast oscillating coefficients [12].

Another method called *FD-HMM* (Finite Difference Heterogeneous Multiscale Method) [1] has offered acceptable numerical results, with reduced computational cost and different scale coefficients. It overcomes the concept of classical homogenization as a method for solving more general problems, with time dependent coefficients.

Later on, the *MsFVM* (Multiscale Finite Volume Method) [13] was introduced to solve elliptic problems arising from flows in porous media, involving different spatial scales. It consists of a pre-processing stage needed to first create the base functions, in which the heterogeneities of the permeability field considered in the problem are captured. This method has raised more developments, called Adaptive Fully Implicit *MsFVM* [14] and Iterative MsFVM [10].

Almost simultaneously, the *FVMsFEM* (Finite Volume Multiscale Finite Element Method) [17] was reported for solving large underground water flow problems in heterogeneous

porous medium problems at different scales. This method is based on a coupling between the finite volume discretization and the finite element multiscale base functions, by capturing the large scale solution in the coarse mesh structure without solving all the small details at the fine mesh. This method estimates the macroscopic flows through control volume segments, bringing the fine scale porous medium data to the larger scales by using base functions that were previously calculated by the finite element method. Such strategie has made this method an available candidate for massive parallel programming.

Another proposal for solving such problems is the *MMMFEM* (Multiscale Mortar Mixed Finite Element Method) [9], applying a similar idea of the domain decomposition in a serie of subdomains. This method computes firstly base functions in local problems as a pre-processing stage. After, considering boundary conditions and source terms, it solves local problems performing linear combinations of the base functions. It is important to remark that this method employs an algorithm without overlapping the domain decomposition.

The recent *FDHMM* (Finite Difference Heterogeneous Multiscale Method) [8] has a new scheme for random porous media which also considers the pre-processing stage.

A hierarchical approach of two scales incorporating post-processing stages is found in the *MuMM* (Multiscale Mixed Method) [7]. This method is based on the nonoverlapping domain decomposition, in which a mixed finite element method is employed to solve second order elliptic problems. The use of base functions reinforces the potential of the multiscale method for the combined parallel programming with the idea of taking advantage of several *CPU-GPU* clusters.

In recent years, the multiscale methods have provided reasonable computational savings for investigating physical phenomena that present wide ranges of heterogeneity scales. The continuous development of computational architectures and *GPU's* improvement together with new software generations have strongly influenced the recent numerical methods and applied scientific programming researches, because it is precise in the exploitation of these newer resources where the state of the art can be found.

In this paper, we introduce another multiscale method based on an overlapping domain decomposition procedure that employs a finite difference discretization to approximate heterogeneous porous medium flow problems. The domain decomposition is employed to solve iteratively global problems at coarse scale. In this method, we also define multiscale basis functions in order to obtain discrete solutions of a family of local problems. The multiscale method takes advantage of the multiple scale, and the family of local basis functions allows to approximate large global problems, efficiently.

This paper is organized as follows. In the second section the governing equation of the problem is presented; a short description of the approximating technique for solving the problem is found in the third section; in the fourth, comparisons between numerical results are presented; and conclusions appear in the last section.

2 GOVERNING EQUATIONS

For simplicity in presentation, it is considered a bounded domain $\Omega \subset \Re^2$, with Lipschitz boundary $\partial\Omega$. The global problem for single-phase fluid, incompressible flow in porous media [4] is given by

$$\nabla \cdot \boldsymbol{u} = f \tag{1}$$

and

$$\boldsymbol{u} = -\boldsymbol{K}\nabla p\,, \tag{2}$$

where $\boldsymbol{K}$ is the permeability tensor divided by the viscosity of the fluid, $\boldsymbol{u}$ is the Darcy velocity, p is the pressure of the fluid, and f is the source/sink term.

Substituting the Eq. (2) into (1), we have a second order linear elliptic equation given by

$$\nabla \cdot (-\boldsymbol{K} \nabla p) = f \ . \tag{3}$$

Typical boundary conditions occuring in porous medium flow problems consist of Dirichlet and Neumann types, respectively written as

$$p = p_b \text{ on } \Gamma_D \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_N \,, \tag{4}$$

where $\boldsymbol{n}$ is an outward unit vector normal to $\partial\Omega$.

The Fig. 1 shows a schematic description of the boundary conditions, based on a cell-centered discretization.

3 THE MULTISCALE FINITE DIFFERENCE METHOD

We have developed the Multiscale Finite Difference Method (*MsFDM*) based on the overlapping domain decomposition procedure to address the global problem, and the multiscale basis functions defined by finite differences to address local problems. The *MsFDM* aims to the fast and accurate approximation of porous medium flow problems that can take advantage of parallel processing units.

3.1 Domain decomposition

A natural way to solve a second order linear elliptic equation in parallel is to divide the domain. In each subdomain, local problems are solved in parallel [5]. The major difficulty in such scheme involves imposing interface constraints between subdomains [15]. We overcome this difficulty working with overlapping subdomains.

The domain Ω is divided into subdomains $\{\Omega_j\}$, $j = 1:M$, where M is the number of subdomains. This domain decomposition produces the coarse grid on Ω. In each subdomain, a local problem is to be solved in the fine grid. Thus, a hierarchical two-scale configuration is posed in the computational domain.

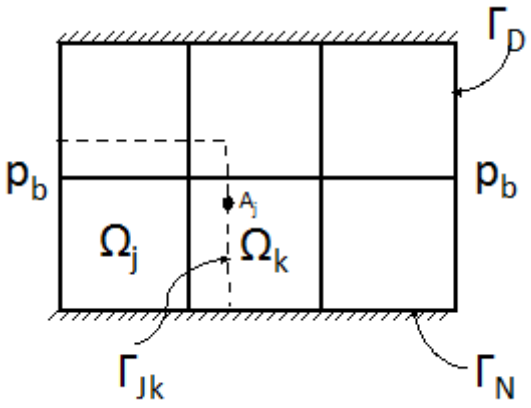


Figure 1: The boundary conditions of the problem

In each subdomain Ω_j, a local problem is given by

$$\nabla \cdot (-\boldsymbol{K}_j \nabla p_j) = f_j \,, \tag{5}$$

where we seek local solutions for the pressure p_j. Denote the single boundary of the subdomain Ω_j, neighbor to the subdomain Ω_k, by $\Gamma_{jk} = \partial\Omega_j \cap \Omega_k$. The boundary condition for a local problem is written as

$$p_j = p_b \text{ on } \partial\Omega_j \cap \Gamma_D, \quad \boldsymbol{u}_j \cdot \boldsymbol{n}_j = 0 \text{ on } \partial\Omega_j \cap \Gamma_N \text{ and } p_j = A_j \text{ on } \Gamma_{jk}. \tag{6}$$

where $\boldsymbol{n_j}$ is an outward unit vector normal to $\partial\Omega_j$ and A_j is a pressure value that comes from neighboring subdomains.

Instead of solving the local problems (5) directly, we define local multiscale basis functions, and represent the discrete solutions by linear combination of such basis functions.

3.2 The multiscale basis functions

In this subsection, we present the *MsFDM* schemes that allow for incorporating fine-scale information of permeability into the coarse-scale iteration procedure.

The refined-scale scheme. First, we define local multiscale basis functions for each subdomain [9]. Considering the subdomain Ω_j, denote the basis functions by ψ_{ji}, $i = 1,...,4N$, that are defined for distinct boundary-value configurations, where N is the number of nodal points along a single boundary Γ_{jk}. The basis functions for the subdomain Ω_j are obtained by solving the following problem:

$$\nabla \cdot (-\boldsymbol{K}_j \nabla \psi_{ji}) = f_j, \tag{7}$$

in which the boundary condition is given by

$$\psi_{ji} = \psi_b \text{ on } \Gamma_{jk}, \tag{8}$$

with $\psi_b = 1$, on the *i*-boundary nodal point, and $\psi_b = 0$, otherwise. The Fig. 2 illustrates boundary-value configurations with $N = 4$ for computing basis functions ψ_{ji}, $i = 1,...,16$.

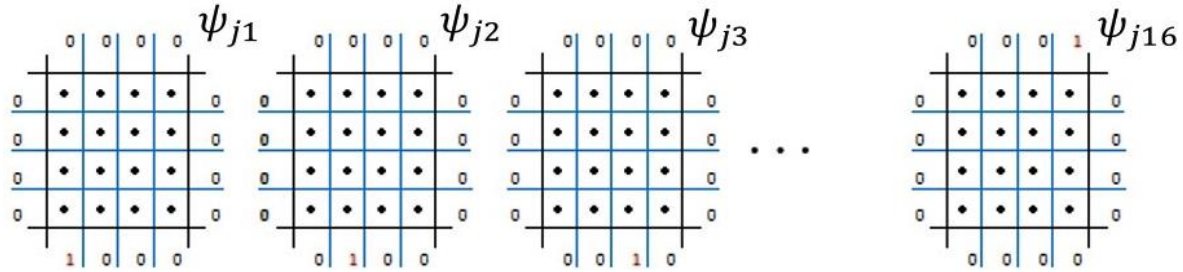


Figure 2: Boundary-value configurations for computing basis functions

The local multiscale basis functions are computed by employing finite differences in the discretization of the problems (7), and using preconditioned conjugate gradient for solving the resulting algebraic equation system. In the *MsFDM*, the basis functions have to be available before performing an iterative domain decomposition procedure. So, they should be pre-computed and stored at the memory of the machines.

The global problem is addressed by performing the coarse-scale iterative procedure. At present iterations, approximate solutions to the local problem (5) can be obtained by the

following linear combination:

$$p^{\nu}{}_{j} = \sum_{i=1}^{4N} A_{ji}^{\nu-1} \psi_{ji} \quad , \tag{9}$$

where the coefficient $A_{ji}^{\nu-1}$ is a pressure value on the *i*-boundary nodal point, at previous iterations. The Fig. 3 shows an illustration of the boundary condition for local problems with $N = 4$.

By the linear combination, there is no linear algebraic problem to be solved in each iteration. In this way, approximate solutions to local problems is obtained without solving them directly.

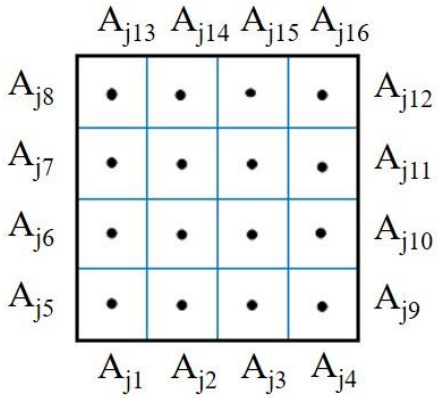


Figure 3: The boundary condition for local problems with $N = 4$

The intermediate-scale scheme. In the refined procedure, we have to compute *4N* basis functions for each subdomain. In order to reduce the number of basis functions we introduce an intermediate scale $\overline{H}$, $h \le \overline{H} \le H$, where *h* and *H* are respectively the fine and coarse-grid steps. Thus, basis functions $\overline{\psi}_{ji}$ must be computed considering the intermediate scale, so as coefficients $\overline{A}_{ji}$ are computed by averaging the pressure.

A balance between numerical accuracy and numerical efficiency can be determined by the choice of the intermediate scale. In the extreme case $\overline{H} = h$, the global problem at fine scale is retrieved, for example.

4 NUMERICAL RESULTS

We perform experiments of slab geometry that are convenient for simulation of single-phase flow problems. The physical domain has $12800 \times 12800 m^2$. The left and right boundary conditions are of Dirichlet type with $p_b = 1{,}0$ Pa and 0,0 Pa, respectively. The top and bottom boundary conditions are of Neumann type with no flow. There is no source term $f = 0$. This slab geometry problem is illustrated in the Fig. 4.

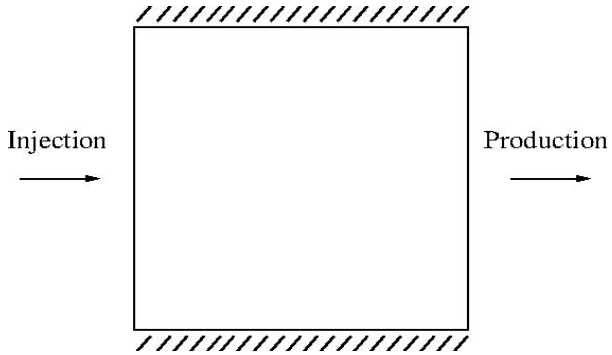


Figure 4: The slab geometry problem

The heterogeneous permeability fields are produced through stochastic realizations based on the relation $K = K_0 \exp(\delta\varepsilon)$, where ε is a self-similar field (Gaussian) characterized on the reference [3], $K_0 = 1.0 \times 10^{-11} m^3 s/kg$, and δ is chosen in order to set a highly permeability ratio.

In these experiments, we consider the extreme case $\overline{H} = H$, where the coefficients $\overline{A}_{ji}$ are computed averaging pressures on the entire single boundary. We perform comparisons of the *MsFDM* solutions with other solutions obtained by using a mixed finite element combined with an iterative domain decomposition method (*MFEM*) [6]. The *MFEM* is an accurate numerical solver. Global errors are evaluated for different permeability ratios K_{max}/K_{min} and coarse grids of 8×8 and 64×64.

Table 1 – Global errors for *MsFDM*

K_{max}/K_{min}	Global Error	
	8×8	64×64
10^1	2.44E-02	1.57E-02
10^3	6.04E-02	3.27E-02
10^6	1.29E-01	4.85E-02

In Table 1, the results for comparisons of global error are shown. As the permeability ratio increases, the global error presents larger magnitude. We can note that the global error is reduced when considering the more refined coarse grid of 64×64. Results of pressure field for *MFEM*, *MsFDM* 8×8 and *MsFDM* 64×64, considering the permeability ratio of 10^6, are depicted in Fig. 4. Following the flow from left to right, a good agreement between pressure curves of *MFEM* and *MsFDM* 64×64 is observed.

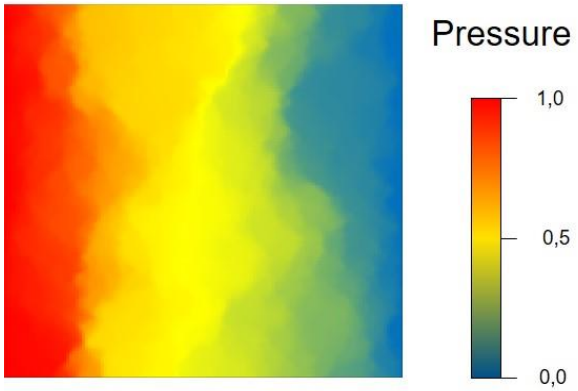
In Table 2 we can find the results of number of iteration and execution time under a coarse grid refinement in the *MsFDM*, considering the permeability ratio of 10^6. As the coarse grid is refined, the number of iteration and execution time increase significantly.

Table 2 – Computational performance in coarse grid refinement

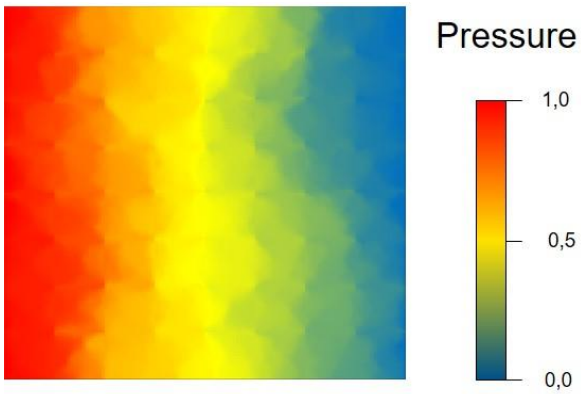
Coarse grid	Iteration number	Execution time (s)
8×8	1391	4,04
64×64	24530	61,71

Simulations of transport contaminant, for example, in this flow field would require a very

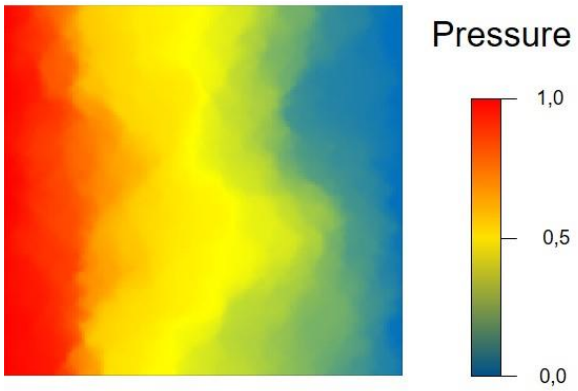
accurate velocity field. The *MFEM* solver is able to compute accurate flow velocity fields even for highly heterogeneous media. Now, we present the results of flow velocity field for comparison between *MFEM* and *MsFDM* 64×64, considering the permeability ratio of 10^6. In the Fig. 5 we can see the flow velocity fields illustrated in arrow fashion, overlapping the permeability fields. Again, we note a good agreement in these results.



(a)



(b)



(c)

Figure 4: The pressure field for (a) *MFEM*, (b) *MsFDM* 8×8, and (c) *MsFDM* 64×64

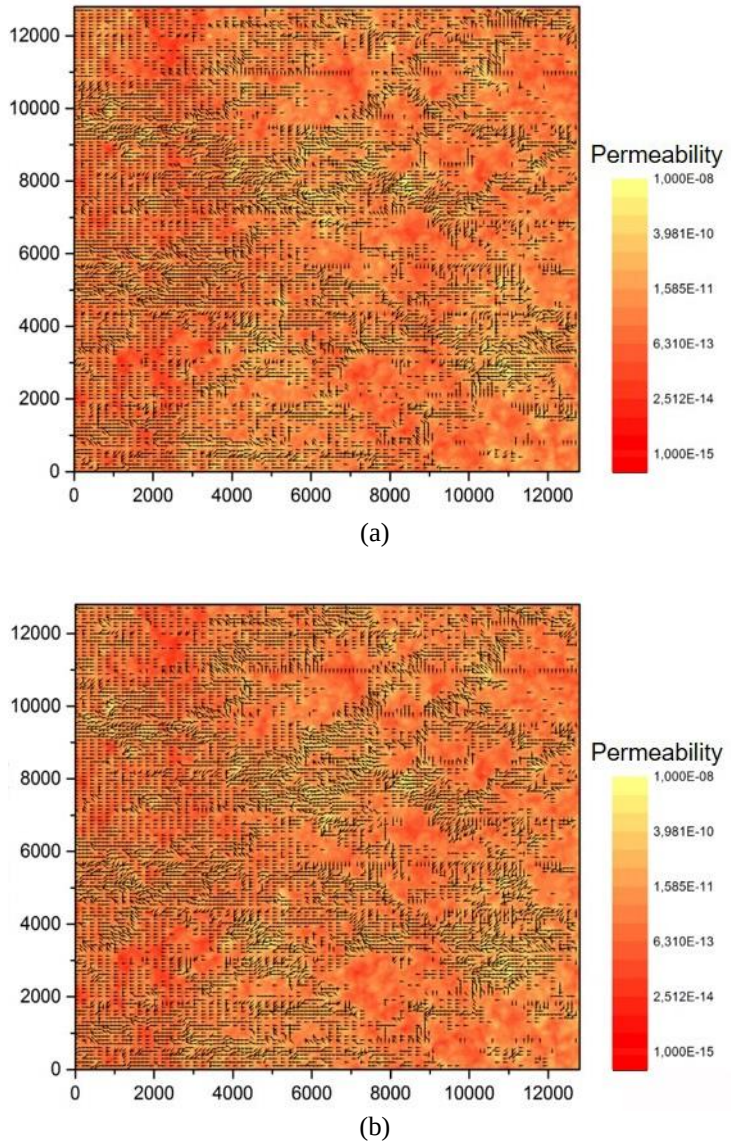


Figure 5: The flow velocity field for (a) *MFEM* and (b) *MsFDM* $64 \acute{} 64$

5 CONCLUSIONS

Despite the comparisons between *MsFDM* and *MFEM* solvers have shown a good agreement in the results, performance measures such as iteration number and execution time are to be improved. We suggest to implement a *SSOR* preconditioner in the gradient conjugate procedure in order to obtain more computational efficiency.

There is a high computational cost in computing the local multiscale basis functions. But this computation is performed at once, and the basis functions are stored at the memory. Therefore, we consider this procedure as a pre-processing stage. That is, this computational cost should not be accounted into the cost of the domain decomposition iteration procedure.

The *MsFDM* is presented as a solver for linear elliptic problems. To accomplish non linear problems, updated local multiscale basis functions should be computed at each iteration as the permeability field is changed by new pressure field. In this situation, the computational cost of computing the local basis functions is increased.

The *MsFDM* fits well in heterogeneous processing units (*CPU-GPU*), and is naturally

adaptable for massively parallel computers, providing better performance.

Acknowledgements

We would like to thank the *CAPES* for supporting the development of this research through the scholarship provided by the *PGMEC/UFF*.

REFERENCES

[1] A. Abdulle, and E. Weinan. Finite Difference Heterogeneous Multiscale Method for Homogenization Problems. *Journal of Computational Physics*, 191, 18-39, 2003.

[2] I. Babuska. Homogenization and its applications, mathematical and computational problems. *Numerical Solutions of PDE*, 3, 89-116, 1976.

[3] M. R. Borges, F. Furtado, F. Pereira, and H.P. Amaral Souto. Scaling analysis for the tracer flow problem in self-similar permeability fields. *Multiscale Model. Simulat.*, 7 (3), 1130–1147, 2008.

[4] W. Chen, M. Gunzburger, F. Hua, and X. Wang. A parallel robin-robin domain decomposition method for the stokes-darcy system. *SIAM J. Numer. Anal.*, 49, 1064-1084, 2011.

[5] C. N. Dawson, Q. I. Du, and T. F. Dupont. A finite difference domain decomposition algorithm for numerical solution of the heat equation. *Mathematical of Computation*, 57 (195), 63-71, 1991.

[6] J. Douglas Jr., P. J. Paes Leme, J. E. Roberts, and J. Wang. A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods *Numer. Math*, 65, 95-108, 1993.

[7] A. Francisco, V. Ginting, F. Pereira, and J. Rigelo. Design and implementation of a multiscale mixed method based on a nonoverlapping domain decomposition procedure. *Mathematics and Computers in Simulation*, 99, 125-138, 2014.

[8] C. Fulai, and R. Li. New scheme of finite difference heterogeneous multiscale method to solve saturated flow in porous media. *Water Resourses Research An Agu Journal*, 44, 2008.

[9] B. Ganis, and I. Yotov. Implementation of a mortar mixed finite element method using a multiscale flux basis. Comput. Methods Appl. Mech. Eng., 198, 3989-3998, 2009.

[10] H. Hajibeygi, G. Bonfigli, M. A. Hesse, and P. Jenn. Iterative multiscale finite-volume method. *Journal of Computational Physics*, 277, 8604-8621, 2008.

[11] T. Y. Hou, and X. H. Wu. A multiscale finite element method for elliptic problems in composite materials and porous media. *Journal Of Computational Physics*, 134, 169-189 1997.

[12] T. Y. Hou, X. H. Wu and Z. Cai. Convergence of a Multiscale Finite Element Method for Elliptic Problems with Rapidly Oscillating Coefficients Mathematics of Computation. *Mathematics Of Computation*, 68 (277), 913-943, 1999.

[13] P. Jenny, S. H. Lee and H. A. Tchelepi. Multiscale Finite Volume Method for Elliptic Problems in Subsurface Flow Simulation. *Journal of Computational Physics*, 187, 47-67 2003.

[14] P. Jenny, S. H. Lee and H. A. Tchelepi. Adaptive fully implicit multi-scale finite volume method for multi-phase flow and transport in heterogeneous porous media. *Journal of Computational Physics*, 217, 627-641, 2006.

[15] T. Min and D. Yang. Parallel finite difference schemes for heat equation based upon overlapping domain decomposition. *Applied Mathematics and Computation*, 186, 1276-1292, 2007.

[16] V. Kippe, J.E. Aarnes, and K.-A. Lie. A comparison of multiscale methods for elliptic problems in porous media flow. *Comput. Geosci.*, 12 (3), 377-398, 2008.

[17] H. Xinguang, and R. Li. Finite volume multiscale finite element method for solving the groundwater flow problems in heterogeneous porous media. *Water Resourses Research An Agu Journal*, 41, 2005.