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A MORE EFFICIENT MATHEMATICAL APPROACH TO THE PORTFOLIO OPTIMIZATION PROBLEM APPLIED IN THE BRAZILIAN STOCK MARKET

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Abstract. This article's objective is to solve a portfolio optimization problem. This portfolio, supposed to be maximized, contains all 67 stocks listed in the Ibovespa, the main index of the Brazilian stock market. The mathematical method presented derives from the Markowitz mean-variance efficient frontier, and was developed by Michael J. Best. This method diminishes the quantity and the complexity of data to be processed and the necessity for advanced computational resources, simply using the regular calculus and the matrix theory. This article proves that Markowitz mean-variance based on diversification, developed almost sixty years ago, still remains the most efficient method for investment decision and asset allocation. The portfolio was optimized for the first semester of 2011, and the goal was to find the global minimum variance portfolio. Results show that the optimized portfolio beat the market in mentioned period and overcame the results of the best domestic investment funds.

1. INTRODUCTION

Almost sixty years ago, an University of Chicago'sgraduate student in economics, while in search of a dissertation topic, ran into a stockbroker who suggested him to study the stock market. Harry Markowitz took that advice and developed a theory that became a foundation of financial economics and revolutionized investment practice. His work earned him a share of 1990 Nobel Prize in Economics [4].

The Markowitz mean-variance diagram plays a central role in the development of theoretical finance. In setting the foundation for the capital asset pricing model, it represents the beginning of modern portfolio theory. Prior to Harry Markowitz's contribution, the field of finance relied much less on mathematical technique. Contributions to the literature tended to be descriptive, or involved only simple operations applied to accounting data. Markowitz's mean-variance paradigm, summed up succinctly in his famous diagram, set finance on the path to becoming a technical scientific discipline, more a branch of economics than of business administration.

The Markowitz diagram considers that all the information about a risky assets' portfoliothat is relevant to investor's aversion to risk can be summed up in on two parameters' values: the standard deviation and the expected value of the portfolio's return, briefly stated as the risk and return.

Markowitz developed mean-variance analysis by selecting a portfolio of common stocks. Over the last decade, mean-variance analysis has been increasingly applied to asset allocation – selection of a portfolio of investments where each component is an asset class rather than an individual security. In many respects, asset allocation is a more suitable application of mean-variance analysis than stock portfolio selection. Mean-variance analysis requires not only knowledge of the expected return and standard deviation on each asset, but also the correlation of returns for each and every pair of assets. Whereas a stock portfolio selection problem might involve hundred of stocks (and hence thousands of correlations), an asset allocation problem typically involves a handful of asset classes (for example, stocks, bonds, cash, real estate, and gold). Furthermore, the opportunity to reduce total portfolio risk comes from the lack of correlation across assets. Since stocks generally move together, the benefits of diversification within a stock portfolio are limited. In contrast, the correlation across asset classes is usually low and, sometimes, negative. Hence, mean-variance is a powerful tool in asset allocation for uncovering large risk reduction opportunities through diversification [4].

The article will be presented as follows: Section 1 describes the properties and the most popular method that can be used to calculate the efficient frontier under the assumption that short sales are always allowed. Section 2 exposes the mathematical method developed by M. J. Best [2], method considered by the author more practical and efficient than presented in section 1. Section 3 applies both methods in Brazilian stock market and summarizes the performance presented by both methods. The last section concludes the article and shows what can be done in terms of constraints to apply the mean-variance efficient frontier on the real investment world.

2. TECHNIQUE FOR CALCULATING THE EFFICIENT FRONTIER

We will derive the efficient set when short sales are allowed and a riskless asset exists, although we perform the calculations with Risk-free asset equal to zero, afterwards in the next section it will be explained. The existence of a riskless asset implies that there is a single portfolio of risky assets that is preferred to all other portfolios. The efficient set is determined by finding that portfolio with the greatest ratio of excess return $(\mu_p + R_f)$ to standard deviation σ_p that satisfies the constraint that the sum of the proportions invested in the assets equals one [3]. The objective function to be maximized is

$$\theta = \frac{\mu_p + R_f}{\sigma_p} \tag{1}$$

Subjected to the constraint

$$\sum_{i=1}^{N} X_i = 1 \tag{2}$$

The constraint is substituted into the objective function and then maximizes as in an unconstrained function. After substitution and stating the data in general form we have

$$\theta = \frac{\sum_{i=1}^{N} X_i(\bar{R}_i - R_f)}{\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq 1}} X_i X_j \sigma_{ij}\right]^{1/2}} \tag{3}$$

The problem stated above is a very simple optimization one and can be solved by the standard methods of basic calculus. To find a function's maximum, it takes the derivative with respect to each variable and set it equal to zero and then solve the system of simultaneous equations. S. Benninga [1] shows how to solve it by using matrix theory.

$$\frac{d\theta}{dX_i} = 0 \tag{4}$$

The efficient set is a straight line with an intercept at the risk-free asset and a slope equal to the ratio of excess return to standard deviation, known as Sharpe Ratio, as shown in Fig 1 [7].

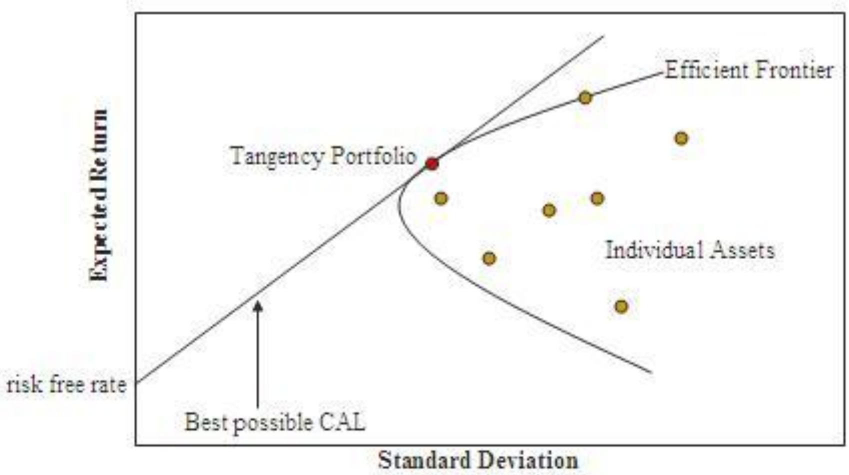

Figure 1: Sharpe ratio

3. A MORE EFFICIENT PORTFOLIO OPTIMIZATION APPROACH

A portfolio consists on various amounts held in different assets. Possible assets' number can be quite large. For example, S&P 500 lists 500 assets of the American financial market. The basic portfolio optimization problem is to decide how much of an investor's wealth should be optimally invested in each asset.

Throughout this section, prime (') will denote transposition. All vectors are column vectors unless primed('). The notation Z_i will be used to denote the i-th component of vector Z.

Let x_i denote the proportion of wealth to be invested in asset i, and let μ_i denote the expected return on asset i, i = 1, . . .n. Let σ_{ij} denote the covariance between assets i and j.
Σ is called the covariance matrix for the assets, which is symmetric and positive semidefinite.

We will make the stronger assumption that Σ is positive definite [5]. In terms of x, the expected return of the portfolio μ_p and the variance of the portfolio σ_p^2, are given by

$$\mu_p = \mu' x \quad and \quad \sigma_p^2 = x'\Sigma x \tag{5}$$

Let l = (1, 1, . . . , 1) $'$; i.e., l is an n vector of ones. Since the components of x are proportions, they must sum to one, i.e., $l'x$ = 1. The constraint $l'x$ = 1 is usually called budget constraint. In the portfolio optimization setting, the goal is to choose a value for x which gives a large value for μ_p and a small value for σ_p^2,. These two goals tend to be in conflict. By introducing the scalar parameter t, the problem has been solved in a nice way. For $t \geq 0$, the parameter t balances how much weight is placed on the maximization of $\mu' x$ (expected return) and minimization of $x'\Sigma x$ (risk) [2].

The most convenient formulation of the objective function to develop the portfolio optimization problem is

$$Min\left\{-t\mu'\mathrm{x} + \frac{1}{2}\mathrm{x}'\Sigma\mathrm{x} \;\middle|\; l'\mathrm{x} = 1\right\} \tag{6}$$

As shown, if t = 0 we will find in Eq.(6) the minimum variance portfolio. As t becomes very large, the linear term in Eq.(6) will dominate and portfolios will be found with higher expected returns at the expense of variance. Consequently, the portfolio becomes more risky.

The optimality conditions for Eq.(6) are

$$t\mu - \Sigma\mathrm{x} = \mathrm{u}l \quad and \quad l'\mathrm{x} = 1 \tag{7}$$

Solving the first equation for X, applying the budget constraint and eliminating u, gives the efficient portfolios as explicit linear functions of t

$$x(t) = h_0 + th_1 \tag{8}$$

Finally we use Eq.(8) to find μ_p and σ_p^2 in terms of t:

$$\mu_p = \mu' h_0 + t\mu' h_1 \tag{9}$$

$$\sigma_p^2 = (h_0 + th_1)' \,\Sigma\, (h_0 + th_1) \tag{10}$$

Part of the value of the efficient frontier is that it condenses information concerning n-assets into a 2-dimensional graphical representation.

On next step, we will apply this method and simulate a 6-months investment period.

4. APPLICATION

The universe of stocks considered on this article is n = 67. These are the stocks that were listed on Ibovespa in 03/01/2011, the main Brazilian stock market index. The assets' expected returns were simply calculated as an average of the past monthly returns and the covariance matrix was constructed as from *betas*, a famous measure of systematic risk. No risk-free asset to perform the calculations was considered, due to the high volatility of the Brazilian interest rates during this period and a riskier market faced by investors in the last three years. So, where the models call for a risk-free asset, it will set equal to zero, reducing the portfolio expected return required.

MatLab® and Microsoft Excel were used to accomplish the following calculations and data was collected in [6] and [9].

Table 1 shows the weights that both methods previously presented gave to each asset. Let ***a*** and ***b*** be the efficient portfolios calculated by the methods in the same order as presented in earlier sections.

Table 1: Efficient portfolio weights

Stock	a	b	Stock	a	b	Stock	a	b	Stock	a	b
ALLL3	0.2%	0.2%	CRUZ3	7.2%	4.4%	ITUB4	6.1%	2.3%	RSID3	1.0%	-0.4%
AMBV4	10.9%	6.9%	CSAN3	-0.1%	0.3%	JBSS3	-2.1%	0.9%	SANB11	-43.9%	-16.9%
BBAS3	5.2%	-0.1%	CSNA3	4.5%	-0.6%	KLBN4	-1.2%	2.7%	SBSP3	2.7%	2.7%
BBDC4	7.2%	2.2%	CYRE3	2.1%	-1.3%	LAME4	3.2%	-0.3%	TAMM4	0.3%	0.3%
BISA3	-3.8%	-0.7%	DTEX3	62.3%	14.3%	LIGT3	2.9%	3.1%	TCSL3	0.5%	0.7%
BRAP4	5.8%	-0.1%	ECOD3	-6.0%	-0.8%	LLXL3	-0.4%	-3.9%	TCSL4	0.0%	0.7%
BRFS3	3.1%	2.5%	ELET3	0.9%	1.8%	LREN3	3.5%	-0.2%	TLPP4	3.7%	10.2%
BRKM5	-2.0%	1.7%	ELET6	1.5%	2.5%	MMXM3	-0.8%	-1.8%	TMAR5	0.0%	3.3%
BRTO4	-0.6%	2.5%	ELPL4	5.5%	4.8%	MRFG3	-4.7%	0.0%	TNLP3	-0.3%	1.7%
BTOW3	-15.3%	-2.7%	EMBR3	-1.6%	2.0%	MRVE3	-0.3%	-2.8%	TNLP4	-0.6%	3.1%
BVMF3	-2.8%	-1.5%	FIBR3	-18.6%	-7.6%	NATU3	5.7%	3.9%	TRPL4	8.2%	4.4%
CCRO3	5.7%	2.8%	GFSA3	-1.4%	-0.4%	OGXP3	0.9%	-1.0%	UGPA4	4.8%	4.6%
CESP6	0.3%	0.3%	GGBR4	0.2%	-1.3%	PCAR4	2.3%	3.3%	USIM3	0.9%	-0.7%
CIEL3	-14.5%	16.9%	GOAU4	-0.2%	-1.2%	PDGR3	3.0%	-0.7%	USIM5	0.3%	-0.8%
CMIG4	4.5%	5.0%	GOLL4	-1.4%	0.5%	PETR3	1.9%	0.7%	VALE3	4.8%	1.5%
CPFE3	7.7%	6.2%	HYPE3	12.6%	3.1%	PETR4	2.0%	0.6%	VALE5	6.4%	2.6%
CPLE6	6.1%	3.6%	ITSA4	9.4%	4.0%	RDCD3	-5.4%	5.7%			

Note the double underlined values. Any asset manager would hold long 0.62 times the capital originally available for investment in only one stock, DTEX3 in this example and sell short an amount equal to 0.439 times the stock SANB11. It bears much risk.

Note that despite these uncommonasset weights, as we can imagine portfolio ***a*** presented higher expected returns than portfolio ***b***,

$$\mu_{pa} = 6.47\% \ \ and \ \ \mu_{pb} = 1.79\%$$

and consequently higher expected standard deviation,

$$\sigma_{pa}^2 = 3.56\% \ \ and \ \ \sigma_{pb}^2 = 1.85\%.$$

The leverage ratio of portfolios ***a*** and ***b*** is 127.75% and 47.64% respectively.

Now we can see why the first method presented in this paper cannot always be applied in practice. It must be taken much attention to sum numbers. The leverage ratio of portfolio ***a*** is not acceptable in some countries or not practiced by asset managers due to its higher exposition to losses and the amount to be borrowed while portfolio ***b*** presents a more distributed proportions between assets and a leverage ratio smaller than 50% which seems more acceptable by practitioners. See the efficient frontier of portfolio ***b*** below in Fig. 2 and note where the global minimum variance portfolio lies.

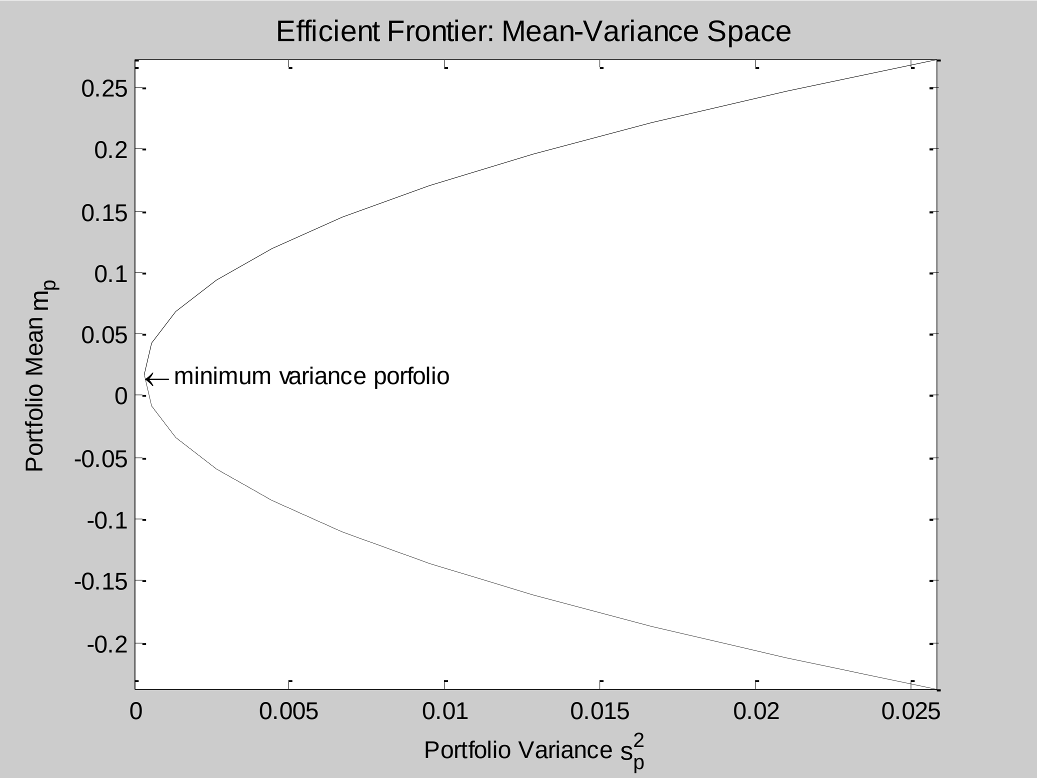

Figure 2: Portfolio ***b*** efficient frontier

Finally we show the results performed in the first semester of 2011 for these two hypothetical portfolios and the comparison with their benchmark and other funds.

The Ibovespa index presented return of -12,38% on the considered period. Portfolio ***a*** returned 4.19% and portfolio ***b*** 18.98%. It means that the feasible portfolio with better distributed proportions, with a smaller leverage ratio and consequently more averse to risk, returns 31.36% above the market.

The most important economic magazine of Brazil [8] publicized in its website the top domestic investment funds ranking of the first semester. Portfolio ***b*** of our study presented superior gross profits. See the table 2 below:

Table 2: Top 7 Brazilian investment funds

1°	PL84	10.3%
2°	Arg	9.4%
3°	OLB	9.0%
4°	ARXLT	8.1%
5°	EOA	6.8%
6°	GAPEV	6.3%
7°	VETFB	6.0%

5. CONCLUSIONS

Mean-variance analysis provides a powerful framework for asset allocation, but as every statistical tool, the mean-variance efficient frontier is subject to estimation errors. Estimation errors or small changes in any input value can dramatically distort the optimization results. Even changes in the correlation between assets over timecan cause an unexpected result.

A better way to treat this market's instability is to rebalance the portfolio over time where inputs should be adjusted to reflect the actual behavior of prices. In order to rebalance the portfolio, we need to resort to the Quadratic Programming.

Another arising problem happens when the short sales are not allowed. By our calculations the unique constraint was the budget one. The imposition of short sales constraints complicates the solution technique, forcing us to resort again to Quadratic Programming.

By the way, Quadratic Programming gives us the choice to incorporate additional constraints to the problem. Once we resort to this technique, it's simple to add other requirements on the solution. For example, we can impose the constraint that limits the minimum dividend yield received or the maximum equity/debt ratio of the portfolio.

But no additional mathematical tool was used to optimize the portfolios showed before and any additional method to choose the stocks (e.g. valuation) was made and even so, portfolio ***b*** presented superb results considering the simple technique that was used to reach the efficient frontier.

Mean-variance efficient frontier has gained wide acceptance among professionals, due to the development of the software packages available to quickly solve the equations with thousands variables. Without support of software packages it would be impossible to accomplish the calculations required, as the Covariance matrix of our study has 4489 cells.

Finally, the emphasis of this article is to show that when investment decision is applied with a knowledge of statistical and mathematical tools, surely it takes great advantage in front of others resources.

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