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NUMERICAL PRICING OF FIXED-INCOME DERIVATIVES

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Abstract. We consider the problem of pricing fixed-income derivatives with the interest rates governed by short rate stochastic processes. We model the financial derivatives via the Feynman-Kac theorem, transforming the conditional expectation problem into a partial differential equation. We then apply a finite difference method to price both first and higher-order derivatives to compare them against closed-form solutions. In the case of Callable bonds, no closed-form formula exists and we compare our results against other numerical method found in the literature. Finally, we engineered some other exotic contracts to extend the results

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1 INTRODUCTION

Fixed-income derivatives are contracts which have the payoff contingent on the evolution of interest rates. The derivatives market have become sophisticated as more complex products aiming to reduce risks appear, complicating the pricing and hedging engines. Our aim is to price interest rate derivatives. We do this twofold:

- via the closed form expressions; and
- via the Modified Full Implicit numerical method found in the literature

With the exception of the Callable bond - which does not have a closed form pricing formula, we compare both approaches for a variety of interest rate derivatives. For the Callable bond price problem, we make a comparison with another numerical method found in the literature. We extend the list to other derivatives that do not exhibit closed form expressions, calculating their prices via the Modified Full Implicit method developed in [4].

We divide the financial instruments into two categories, adopting the derivative's order classification of [16] due to the nature of numerical approximating method via Partial Differential Equations.

Derivative instruments of first order are those whose payoff depends only on the quantity we are directly modeling. A bond or an interest rate cap/floor depends directly on the probabilities of the interest rate. On the other hand, a bond option depends on the price of the bond, which in turn, depends on the interest rate. So the bond option is a second order derivative.

Derivative instruments of second order are computationally more expensive than the first order ones, because we must solve the lower level first and use the results to feed the higher level problems. The numerical price of a bond option, for example, is found by solving - via a PDE and its associated terminal condition - a zero-coupon bond backwards from S to T, where T is the maturity of the option and S is the maturity of the bond. Then, using the bond price at time T as the new terminal condition, we solve again the PDE in $[0, T]$ to get the price of the option at time zero.

Higher order derivatives mean that we have to solve more than one numerical problem. Numerical errors in lower levels can contaminate the higher levels, which could result in prices that are completely different from the fair prices.

Following [2], the economy we consider in this work has the trading interval $[0, T]$. The uncertainty under the real-world probability measure is completely specified by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the complete set of all possible outcome elements $\omega \in \Omega$. The information available is contained within the filtration $\mathcal{F}_{t\geq 0}$, such that the level of uncertainty is resolved over the trading interval with respect to the information filtration. The last term, completing the probability space, is called the real-world probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, since it reflects the probability law of the data.

With the money market account as numeraire, the price of a derivative contract at time t is just the expectation, given all relevant information up to t, of the corresponding terminal payoff $G(r_T)$ in some probability measure $\mathbb{Q}$, equivalent to the probability measure $\mathbb{P}$, called risk-neutral measure. This measure makes the derivative contract a martingale under $\mathbb{Q}$. A detailed explanation about equivalent martigale measures, risk-neutral pricing and other stochastic calculus tools go beyond the scope of this work, but can be found in [2, 12, 10, 13].

From now on, we consider the economy driven by the Vasicek short-term interest rate model [15]. The diffusion process is a mean-reverting version of the Ornstein-Uhlenbeck process. The short-term interest rate process r_t is defined as the unique strong solution of the Stochastic Differential Equation

$$dr_t = a(b - r_t)dt + \sigma dW_t^{\mathbb{Q}} \tag{1}$$

where a, b and σ are strictly positive constants [10]. In Eq. (1), b designates the mean reversion level, a is the reversion speed and σ is the volatility of the short rate. Under the martingale probability measure $\mathbb{Q}$ the process $W^{\mathbb{Q}}$ is a one-dimensional Brownian motion.

The parameters of the model can be historically estimated (see e.g. [3]) or calibrated so that the spot rate model fits the discount factor curve. If the market bond prices are arbitrage-free, so the calibrated interest rate model will be [11].[1]

It can be proved with the use of Ito's formula that the interest rates can be evaluated by

$$r_t = b + e^{-at}(r_0 - b) + \sigma e^{-at} \int_0^t e^{-as} dW_s^{\mathbb{Q}}, \tag{2}$$

where $r_0 > 0$, so that the short-rate process r_t has a conditional Gaussian probability distribution with mean

$$\mathbb{E}^{\mathbb{Q}}\left[r_t | \mathcal{F}_0\right] = b + (r_0 - b)e^{-at} \tag{3}$$

and variance

$$Var^{\mathbb{Q}}\left[r_t | \mathcal{F}_0\right] = \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{4}$$

The market price of risk is assumed constant with $b = b' - \frac{\lambda\sigma}{a}$. Thus, the short-rate process is a process under the martingale measure $\mathbb{Q}$ equivalent to the process $\mathbb{P}$, with a translation of the long run level of the short rate.

The numerical solutions found below are based on the PDE (see the Discounted Feynman-Kac Theorem in Appendix A)

$$\frac{\partial U}{\partial t} + a(b - r)\frac{\partial U}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial r^2} = rU \tag{5}$$

[1]In fact, the Vasicek model can not fit perfectly the bond prices, but an approximation can be done as will be seen later.

with terminal condition

$$U(T,r) = h(r) \tag{6}$$

and boundary conditions specified according to each instrument. It is worth noticing that a preliminary and important step is taken by means of introducing a coordinate transform with respect to the x-axis in PDE Eq. (5) (see [4] for details), before entering with the Modified implicit method.

2 FIRST ORDER DERIVATIVES

2.1 Zero-coupon bonds

Bonds are debt instruments under which the owner receives interest from the issuer in form of coupon payments and/or the difference between the prices in the days of trade and maturity. Commonly issued by governments and corporations to finance their spending and investments, due to its high liquidity, financial engineers use bonds to design other securities, risk managers use them to replicate portfolios, speculators to bet on interest rate changes and central bankers to plan monetary policies.

When a zero-coupon bond is considered free of default risk, its arbitrage-free price under the risk-neutral measure $\mathbb{Q}$ is formulated as

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right], \quad \forall \ t \in [0,T]. \tag{7}$$

Zero-coupon bonds are the main product in the unconditional derivatives class. They serve as a basis for a variety of interest rate derivatives.

The following expression developed by [15] is used to calculate the price at time t of a zero-coupon bond that pays 1 at time T:

$$P(t,T) = \alpha(t,T)e^{-\beta(t,T)r(t)}. \tag{8}$$

In this equation,

$$\beta(t,T) = \frac{1-e^{-a(T-t)}}{a} \tag{9}$$

and

$$\alpha(t,T) = \exp\left[\frac{(\beta(t,T)-T+t)(a^2b-0.5\sigma^2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right] \tag{10}$$

To find the estimates of prices of a zero-coupon bond via finite difference methods we proceed as follows.

We prescribe the terminal condition

$$U(T,r) = NP, \tag{11}$$

where NP is the notional principal of the contract, generally set as 1. We also truncate the infinite domain between r_{min} and r_{max} to establish the boundary conditions

$$U(t, r_{min}) = 0 \tag{12}$$
$$\frac{\partial U(t, r_{max})}{\partial r} = 0. \tag{13}$$

Now, setting the parameters $a = 0.1$, $b = 0.1$ and $\sigma = 0.005$ as an illustrative example, with 800 points in the spatial grid and 5 time-steps per day for a two-year bond, boils down to the prices illustrated in Figure 1. We observe that the solution converges fast to the analytical solution (according to the closed form expressions of [15]) in the spatial domain. We also notice that there are huge left boundary numerical errors apparent in Figure 2. This fact is due to our failure in specifying correctly the boundary condition. This is in fact a source of discussion in financial engineering books and papers like [7] and [5], respectively. We still note in Figure 5 that if we specify the Dirichlet homogeneous boundary conditions, namely

$$U(t, r_{min}) = 0 \quad \text{and} \quad U(t, r_{max}) = 0, \tag{14}$$

or the Neumann homogeneous boundary conditions, i.e.,

$$\frac{\partial U(t, r_{min})}{\partial r} = 0 \quad \text{and} \quad \frac{\partial U(t, r_{max})}{\partial r} = 0, \tag{15}$$

then the numerical solution converges immediately towards the analytical solution is space. Reinforcing our early statement: boundary conditions do not affect the solution in the domain of interest when using the Vasicek model and the truncated domain is reasonably large.

Figure 4 shows the prices for one-year zero-coupon bond considering three different shapes of the term structure (see Figure 3) when $r(t) = 0.09$. An increasing yield curve results in cheaper bond prices and a decreasing yield curve results in more expensive bond prices. Humped yield curve results in prices in the middle, close to the cheaper prices, but depending on the maturity of the bond.

The relative errors are calculated according to the following measure:

$$e = \frac{|\text{numerical} - \text{exact}|}{\text{exact}}. \tag{16}$$

In practice, it is assumed that all market information is available in the interest rates data or quoted liquid instrument prices. So, we need to perform the task of finding the best-fit parameters in a parametric model in comparison with an observed quantity.

Calibration aims to price and hedge non-liquid or non-traded instruments, which does not have much information available in the market. A calibration method seeks the

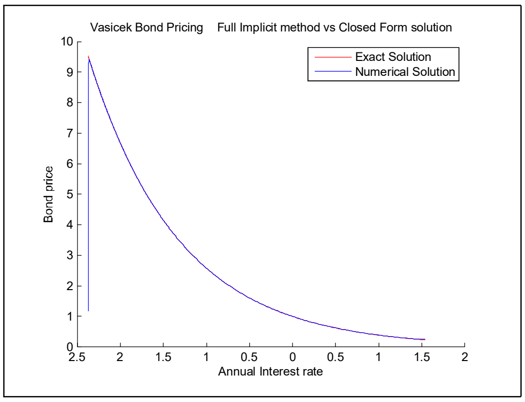

Figure 1: *Zero-coupon Bond Price*

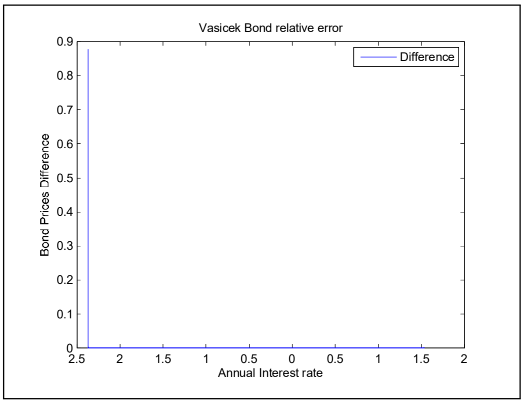

Figure 2: *Pricing relative error*

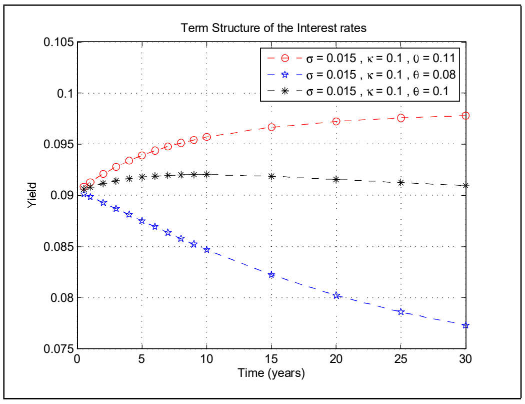

Figure 3: *Term Structures of the Interest Rates*

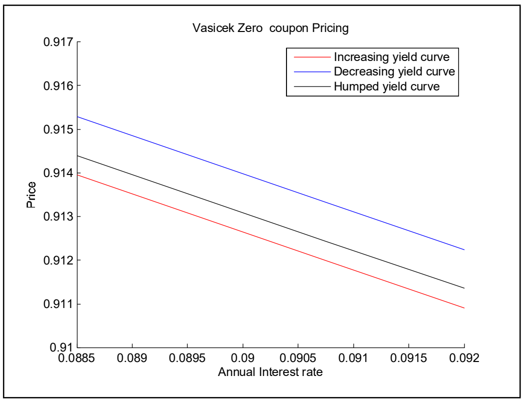

Figure 4: *Comparison of parameters*

model parameters that results in the minimal error between model prices and observed market prices, such as bond or cap prices. These parameters can be used to price callable bonds and exotic options for instance.

Figure 6 below gives us the calibration of the Brazilian 1-month (yearly compounding) Term Structure of the Interest Rates observed on $19/09/2014$ at [1] for the Vasicek model, using the Sequential Quadratic Programing algorithm of [9]. The parameters found were $r_0 = 0.10844, \alpha = 0.35832, \beta = 0.165291$ and $\sigma = 0.18237$. One month later, on $20/10/2014$ the parameters found were $r_0 = 0.11175, \alpha = 0.3948, \beta = 0.171651$ and $\sigma = 0.20463$ with the term structure shown as in Figure 7. If we choose the GMM analysis described by [3] to estimate historically the parameters, the 3-month brazilian interest rates observed between $04/04/2000$ and $19/09/2014$ results in $\alpha = 0.1265, \beta = 0.0802$ and $\sigma = 0.0218$.

As stated by [7], the Vasicek model cannot perfectly fit the yield curve. To this end, a new class of short-rate models have been developed, like models with time-dependent parameters and lognormal dynamics.

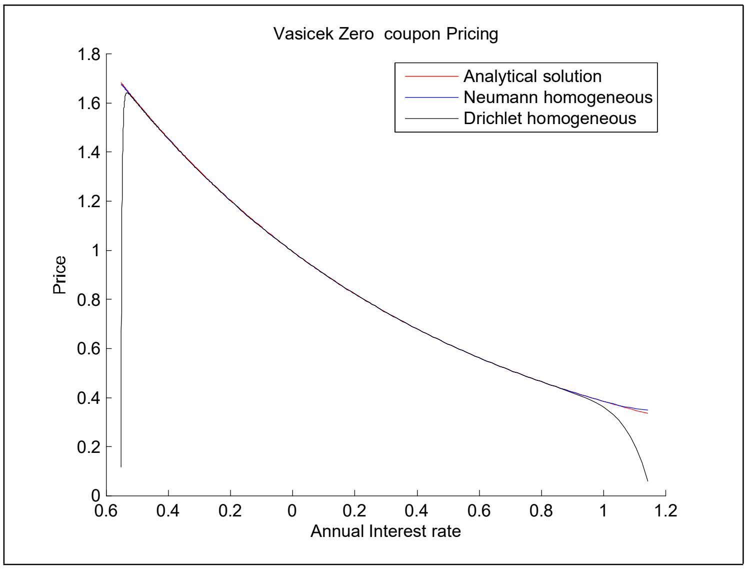

Figure 5: *Comparison of boundary conditions*

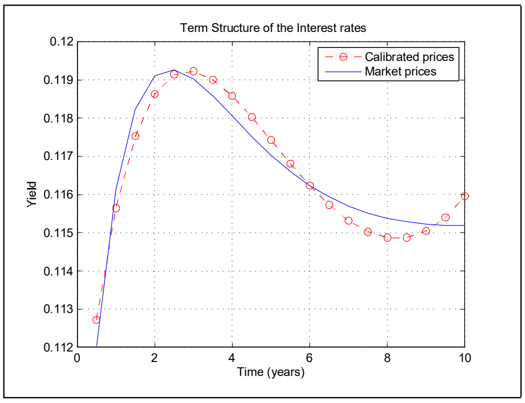

Figure 6: *SQP calibration of the Term Structure of the Interest Rates in Sep/14*

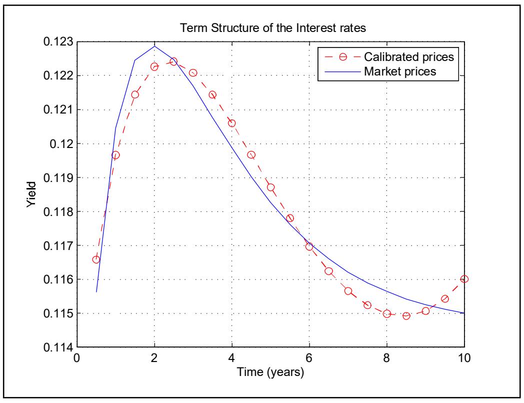

Figure 7: *SQP calibration of the Term Structure of the Interest Rates in Oct/14*

2.2 Coupon bonds

A coupon-bearing bond can be seen as a portfolio of zero-coupon bonds with maturities equal to its associated coupon paying date and face value equal to the coupon rate.

Coupon-bearing bonds which coupons remains constant over time is just the cumulation of payments P_n discounted with the particular zero-bond prices $P(t, T_n)$:

$$P(t, T_N) = \sum_{n=1}^{N} \mathbb{E}^{\mathbb{Q}} \left[P_n e^{-\int_t^{T_n} r_s ds} | \mathcal{F}_t \right] \quad \forall \ t \in [0, T_n]. \tag{17}$$

Instead of solving a set of zero-coupon bonds, which can remarkably increase the computational effort, we just add the coupon value $\bar{K}$ at the due date t_k through the following jump condition [16]:

$$u(r, t_k^-, T) = u(r, t_k^+, T) + \bar{K}(t_k). \tag{18}$$

The terminal condition and the boundary conditions are equal to those of the case of zero-coupon bonds, determined by Eq. (11) with $NP = 1000$ and Eq. (15), respectively. The conclusions are similar to those of zero-coupon bonds as Figures 8 and 9 indicate. It is important to observe the discontinuity caused by the jump conditions in the coupon dates (see Figure 10).

The parameters were $a = 0.5$, $b = 0.1$ and $\sigma = 0.1$, with 1000 points in the spatial grid and 5 time-steps per day for a one-year bond with 10% coupons paid every trimester. A debt issued in this set-up would rise \$1281 per bond if the current short rate is 10.1%.

Henceforward we will restrict the analysis to the domain of interest, i.e. $r \in [-0.05, 0.5]$.

We have also performed numerical experiments with coupons that depend on the level of the interest rates. Or else, the coupon rate is as high (low) as the instantaneous interest rate. The jump condition then becomes

$$u(r, t_k^-, T) = u(r, t_k^+, T) + \breve{K}(r). \tag{19}$$

In this case, an increasing shape (instead of the decreasing shape of the previous case) appears for the pricing solution (see Figure 11).

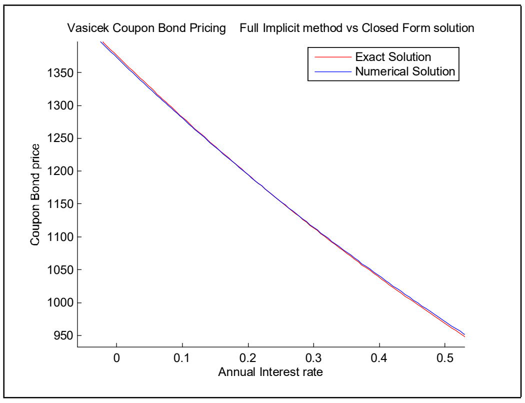

Figure 8: *Coupon Bond Price*

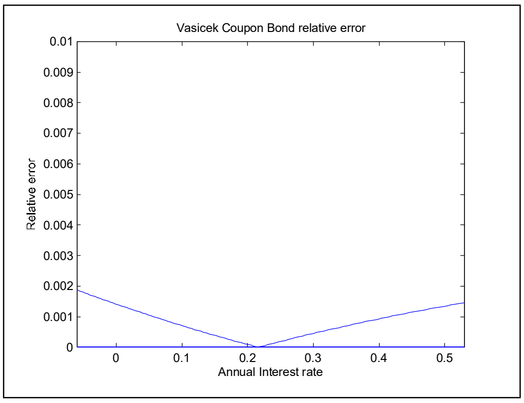

Figure 9: *Pricing relative error*

2.3 Forward-rate Agreements

Forward Rate Agreements are forward contracts on interest rates. A forward interest rate can be derived from expectations, solving the PDE with its appropriate terminal condition, or simply using arbitrage-free arguments over the yield curve.

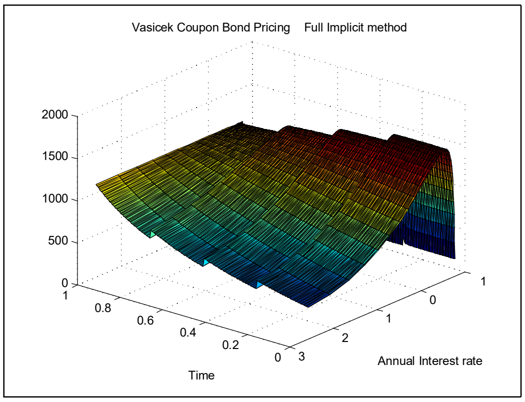

Figure 10: *Coupon Bond Price surface*

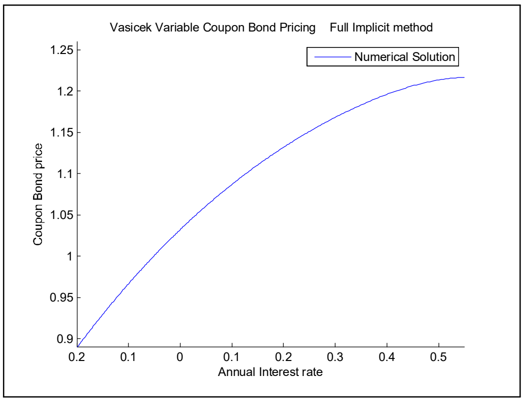

Figure 11: *Variable coupon bond prices*

Let us assume a yearly compounding interest-rate. Then the one-year rate, one year from today, is given by

$$(1 + {}_0R_1)(1 + {}_1R_2) = (1 + {}_0R_2)^2, \tag{20}$$

where

${}_0R_1$	is the one-year interest rate today,
${}_0R_2$	is the two-year interest rate today,
${}_1R_2$	is the forward interest rate for one year, a year from today.

The last expression stands for the rate between years 1 and 2, which are implicit in the yield curve (see [14]).

By pricing zero-coupon bonds of various maturities, we can obtain the continuously compounded term-structure of the interest rates. This is equivalent to a forward curve between today and the corresponding maturity (see the magenta line in Figure 12). The forward rates can be converted into simply compounded ones (see Figure 13).

Let $R(t,t,S)$ denote the continuously compounded today's interest rate with maturity in S. Denote $R(t,T,S)$ the today's interest rate contract from T with maturity in $S > T > t$. Based on arbitrage-free arguments on the yield curve we have that

$$e^{-R(t,t,T)(T-t)}e^{-R(t,T,S)(S-T)} = e^{-R(t,t,S)(S-t)}. \tag{21}$$

By simple algebraic manipulations, the forward rate from T to S reads as

$$R(t,T,S) = -\frac{1}{(S-T)}\left[R(t,t,T)(T-t) - R(t,t,S)(S-t)\right]. \tag{22}$$

For an arbitrarily fixed t, seen as present time, and varying T from 0.5 to 1.5 years we derive the forward curves with maturities S up to 10 years (see Figure (12)). The parameters used in the bond pricing is $a = 0.15$, $b = 0.115$ and $\sigma = 0.03$.

A forward rate agreement is the exchange of a fixed interest rates for a floating interest rate in some prescribed future date. For simply-compounded interest rate, the price of a forward contract can be calculated by looking at the yield curve. Equation (23) below calculates the fixed rate at which the contract's price values zero, that is, the forward rate.

$$\bar{K} = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^S r_s ds}\left(P(T,S)^{-1} - 1\right)|\mathcal{F}_t\right]}{\tau(T,S)P(t,S)} \tag{23}$$

$$= \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{24}$$

Following the same reasoning for continuously compounded interest rates, we obtain that the fixed rate that makes the FRA's price values zero is the forward rate given by Eq. (22).

Let us give an example of the use of FRA inspired in the examples given by [14]:

- Due to unforeseen circumstances the ABC Corp's[2] treasury manager anticipated a need for a 2 year borrow of 10 millions one year from now, for which it would pay an interest rate determined by the then current 2-year continuously compounded floating interest rate. It is important to ABC Corp to lock today in the prevailing yield curve to hedge against the exposure to the 2-year interest rate.

The today's spot 1-year yield is 10.09%, and the today's spot 3-year yield is 10.19%. From Eq. (22) the 2-year forward rate in one year is 10.24%, which means the rate fixed for the 1×2 FRA contract.

In the case the ABC Corp enters in this FRA, it will lock its 10 million debt in 10.24%. The result is:

i) if in 1 year the 2-year interest rate is 0.26% higher than 10.24%, ABC Corp will receive the difference in the FRA and will pay 10.5% in its debt;

ii) if in 1 year the 2-year interest rate is 10%, ABC Corp will pay 0.24% to its FRA counterpart and 10% to the lender.

It is noteworthy that the ABC Corp hedged its loan in 10.24%.

There are cases where the borrower has an income fixed to some interest rate $\hat{K}$ and wants to lock its debt in it to match its assets and liabilities. If $\hat{K}$ differs from the current forward rate, the price of the contract will be given by

$$FRA = N\left[P(t,S)(\tau(T,S)K + 1) - P(t,T)\right] \tag{25}$$

[2] This is a fictitious company.

for simply-compounded interest rate, or by

$$FRA = (S-T)P(t,S)\left[\hat{K} + \frac{\ln(P(t,S))}{S-T} - \frac{\ln(P(t,T))}{S-T}\right] \tag{26}$$

for the continuously compounded interest rate case.

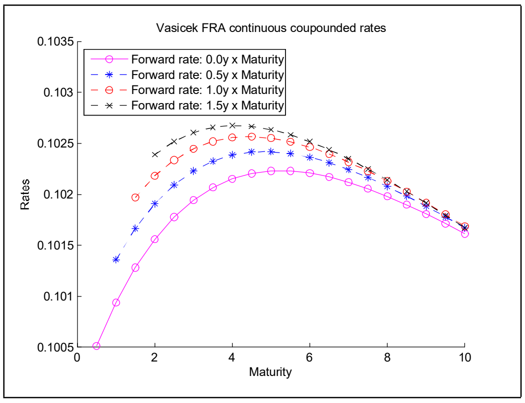

Figure 12: *Continuous compounded FRA*

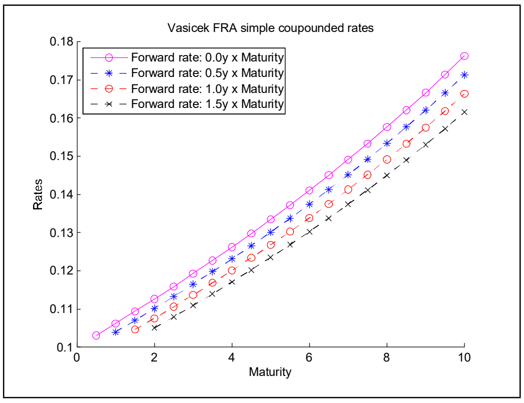

Figure 13: *Simple compounded FRA*

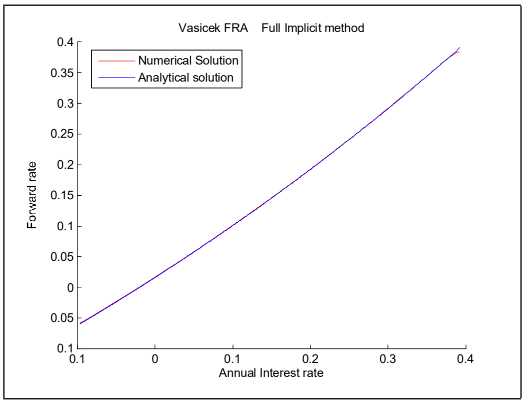

Figure 14: *FRA*

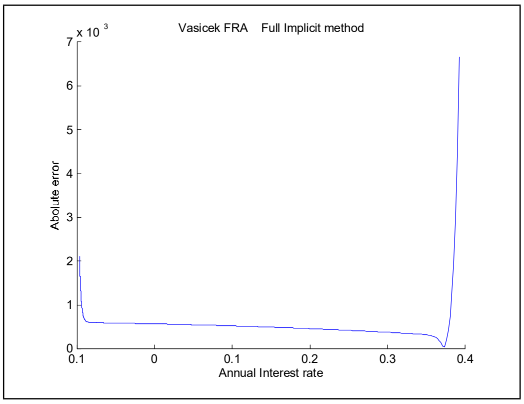

Figure 15: *FRA absolute error*

We end this section stating that, if we can obtain, numerically, the prices of a zero-coupon bonds, we can efficiently solve the FRA prices via arbitrage-free arguments, i.e., via the interest rate term structure.

2.4 Caps, Floors and Swaps

The example shown in Figure 16 (respectively 18) stands for a 1-year option protection against a rise (fall) in the interest rate $r(T)$ above (below) the strike $K = 0.1$. If the caplet - a single cap, is exercised, the PDE terminal condition $(r(T) - K)$ times the principal is paid one year later. In the case of a floorlet - a single floor, if exercised, the contract pays $(K - r(T))$ times the principal one year later. The parameters were set for both examples as $a = 0.2$, $b = 0.12$ and $\sigma = 0.03$.

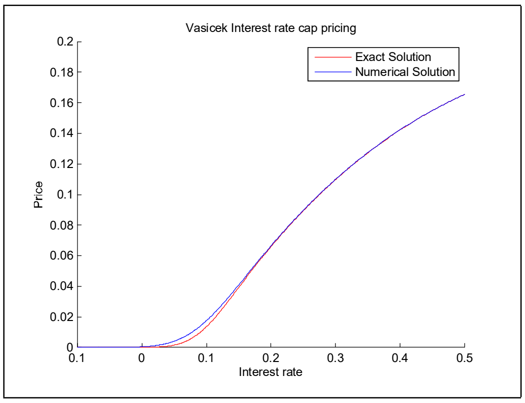

Figure 16: *Interest rate Caplet price*

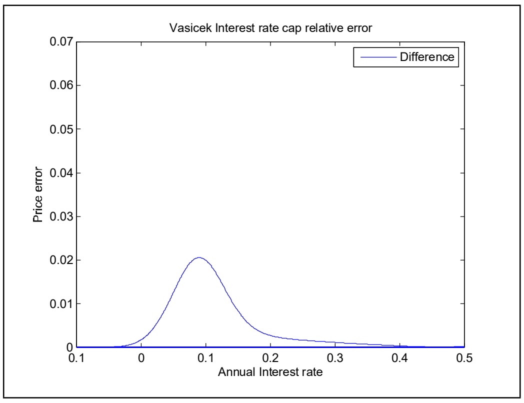

Figure 17: *Pricing relative error*

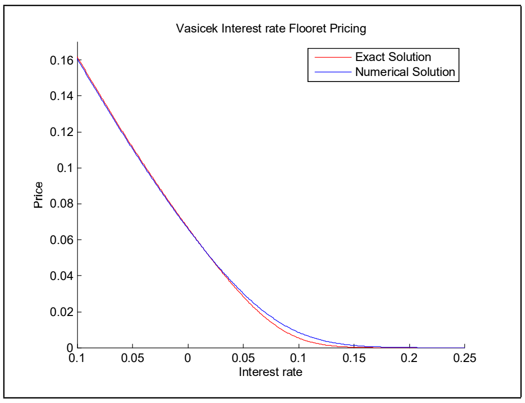

Figure 18: *Interest rate Floorlet price*

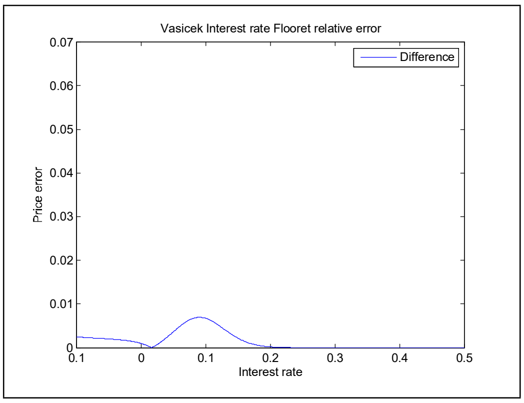

Figure 19: *Pricing relative error*

A caplet/floorlet can be viewed as an option to enter in a Forward Rate Agreement at a specified time T and at a rate K. Since these contracts have options embedded, that is, they are options and not obligations. Even in the case K equals the correspondent forward rate they will not have zero value.

An interest rate cap or floor at a rate K, gives the owner the right to enter in a series of J FRA's every $\frac{T}{J}$ year for T years.

It is worth noticing the following relationship:

$$\text{caplet}_{t,T}(K) - \text{floorlet}_{t,T}(K) = \text{FRA}_{t,T}(K), \tag{27}$$

that is, to buy a cap and sell a floor is equivalent to enter in a Forward Rate Agreement that pays a rate equal to the strike and receives the floating rate. The relation Eq.

(28) expresses the Put-Call parity for caps and floors. It provides us with an arbitrage condition for building and pricing one contract from two of them.

A Swap is a series of FRA's. Consequently,

$$\text{cap}_{t,s_j,T}(K) - \text{floor}_{t,s_j,T}(K) = \text{Swap}_{t,s_j,T}(K), \tag{28}$$

Figure 20 shows the solutions for single contracts, where the 1×2-year swap prices are calculated from the put-call parity analytically and numerically. The parameters are set as $a = 0.2$, $b = 0.12$ and $\sigma = 0.03$. The fixed rate for the swap (or equivalently, the strike for the cap and floor) is set as 10%.

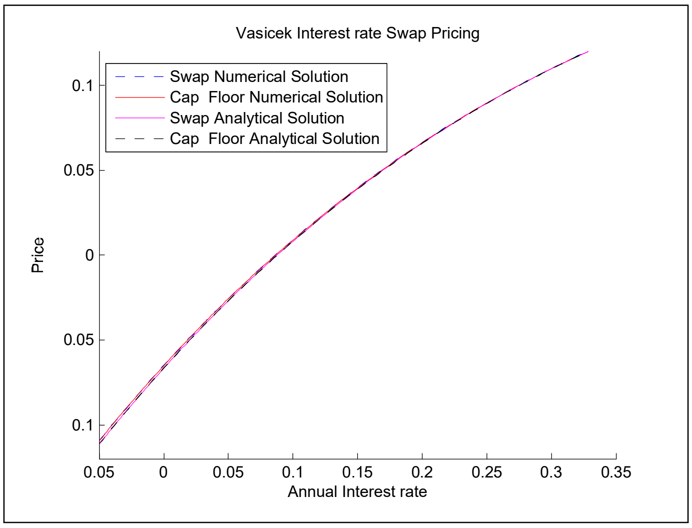

Figure 20: *Swap by put-call parity*

3 HIGHER ORDER DERIVATIVES

3.1 Bond Options

Bond options belong to the higher order derivatives class. Recalling, a zero-coupon bond call (put) option gives its holder the right, but not the obligation, to buy (sell) a zero-coupon bond $P(t, S)$ for a predetermined strike price K at time $T \in (t, S)$. Thus we must find the price at time T of the zero-coupon bond with maturity in S, and then use this solution in the terminal condition

$$U(T, r) = \max(P(T, S) - K, 0) \tag{29}$$

for the option's PDE, that runs from T to t.

The price of a zero-coupon bond call option can be obtained by solving

$$ZCB_c = N\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \max\Big(P(T, S) - K, 0\Big)|\mathcal{F}_t\right] \quad \forall \ t \in [0, T]. \tag{30}$$

[8] found the closed-form solution to the above conditional expectation problem so that the zero-coupon bond call option can be directly calculated by

$$ZCB_c = NP(t,S)\Phi(d_1) - KP(t,T)\Phi(d_2), \tag{31}$$

where

$$d_1 = \frac{ln\left(\frac{NP(t,S)}{KP(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \tag{32}$$

$$d_2 = d_1 - \sigma_p \tag{33}$$

and

$$\sigma_p = \sigma\left(\frac{1-e^{-a(S-T)}}{a}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}. \tag{34}$$

The function $\Phi(d)$ corresponds to the probability of a standard normal random variable being less than d, namely

$$\Phi(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \tag{35}$$

The parameters of the numerical simulation were set as $a = 0.1$, $b = 0.1$ and $\sigma = 0.02$, with 1000 points in the spatial grid and 5 time-steps per day for a one-year option in a two-year zero-coupon bond with strike $K = 0.8$. Again, the conclusion is similar to those of the last section, as can be seen in Figures 21 and 22: we notice in these figures that the errors at the left boundary does not contaminate the solution. Furthermore, in the case these errors contaminate the solutions - which could be perhaps the case of other numerical methods, then the solution would deteriorate because the same numerical scheme would be applied twice.

The relative errors are calculated according to the following measure:

$$e = \frac{|\text{numerical} - \text{exact}|}{\text{Principal}} \tag{36}$$

Figure 23 shows the surface that relates the prices of the zero-coupon bonds from its maturity to the option's maturity, against interest rates and time. We can clearly see that the solution is spurious oscillation free. We also see in Figure 24 that the level curves exhibit very high bond durations in the extreme negative interest rates region.

For risk management purposes, zero-coupon bond put options can be viewed as caplets, protecting the owner against the interest rate rise. Conversely, zero-coupon bond call options can be viewed as floorlets, protecting the owner against the interest rate fall.

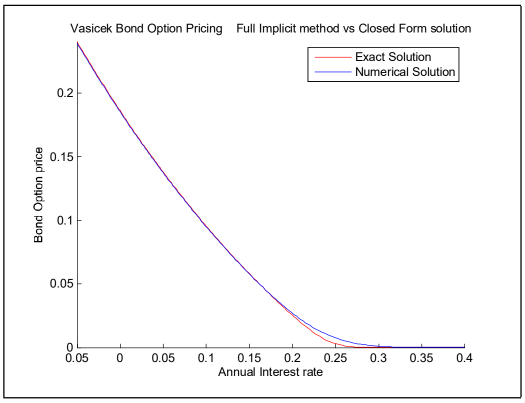

Figure 21: *Zero-coupon Bond Option Price*

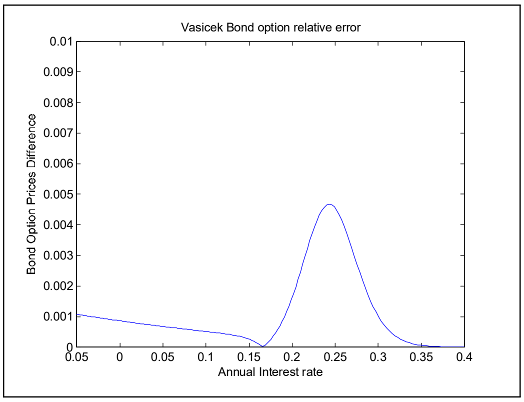

Figure 22: *Zero-coupon Bond Option Pricing Error*

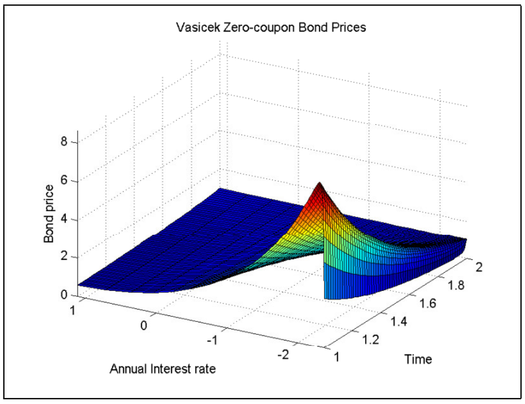

Figure 23: *Zero-coupon Bond Pricing solution surface*

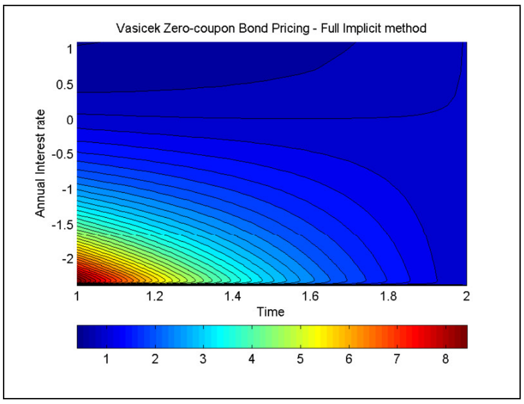

Figure 24: *Zero-coupon Bond Pricing levels*

The price of a zero-coupon bond put option can be obtained as

$$ZCB_p = N\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\max\Big(K - P(T,S), 0\Big)|\mathcal{F}_t\right] \quad \forall\ t \in [0,T]. \tag{37}$$

Then the zero-coupon bond put option can be directly calculated by

$$ZCB_p = KP(t,T)\Phi(-d_2) - NP(t,S)\Phi(-d_1) \tag{38}$$

A put-call parity can be straightforward developed for pricing bond put options from calls. At option maturity T we have that

$$ZCB_c(T) - ZCB_p(T) = P(T,T,S) - K. \tag{39}$$

In other words, the difference between the payoffs of call and put options on the zero-coupon bond with maturity in S is equal to the difference between the market price of the bond and the strike price K. At present time t we have

$$ZCB_c(t) - ZCB_p(t) = P(t,T,S) - P(t,T,T)K, \tag{40}$$

which refers to the put-call parity for bond options.

3.2 Options on coupon bonds

A closed-form expression for pricing options on coupon-bearing bonds was also developed in [8]. It follows:

The trick consists in a process to transform an option on a coupon-bond into a portfolio of zero-coupon bond options.

Let us consider the price $P(r,t,T_1,...,T_J)$ of a coupon-bearing bond paying J coupons c at $j = 1,...,J$ dates and the following payoff of an option that gives the right to buy this bond paying the strike K at T_0:

$$CBB_c = \max(P(r,t,T_1,...,T_J) - K, 0). \tag{41}$$

To find its price, we first need to solve a nonlinear equation problem to find $\hat{r}$ that makes

$$P(\hat{r},t,T_1,...,T_J) = K. \tag{42}$$

We then match the artificial strikes K_j with each of the unit discount price values in the coupon dates using $\hat{r}$, or else,

$$\bar{K}_j = P(\hat{r},t,T_1,...,T_J), \tag{43}$$

Reminding Eq. (31) the price of the option is given by

$$CBB_c = \sum_{j=1}^{J} c_j(ZCB_c(r,t,T_0,T_j,\bar{K}_j). \tag{44}$$

The numerical price can be found by simply solving the same problem as before, but entering with the coupon-bond price as initial condition of the option problem, instead of the zero-coupon bond.

Figures 25 and 26 illustrate the case of a one-year option that gives the right to buy a 5% semiannual 2-year coupon bond for $\$0.93$. We used a 1000 spatial points grid and with parameters set as $a = 0.1$, $b = 0.07$ and $\sigma = 0.015$.

Figure 27 compares the prices of one-year bond options with the same strike, where we vary the 2-year semiannual coupon size. The conclusion is as expected: the price of the options decreases as the coupon payments does. The zero-coupon bond option is the cheapest one.

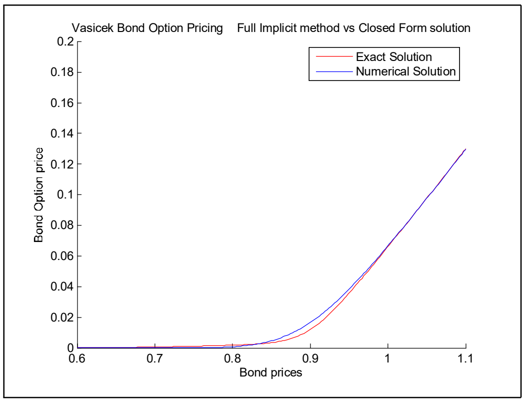

Figure 25: *Coupon bond option pricing*

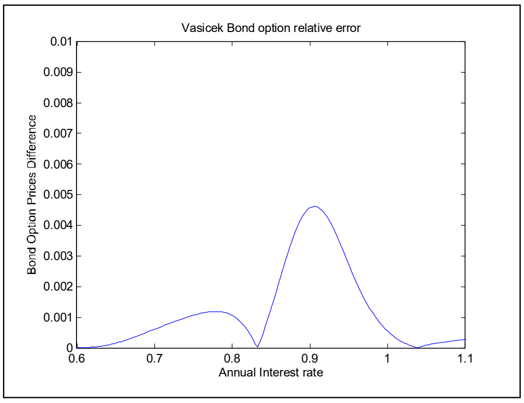

Figure 26: *Coupon bond option error*

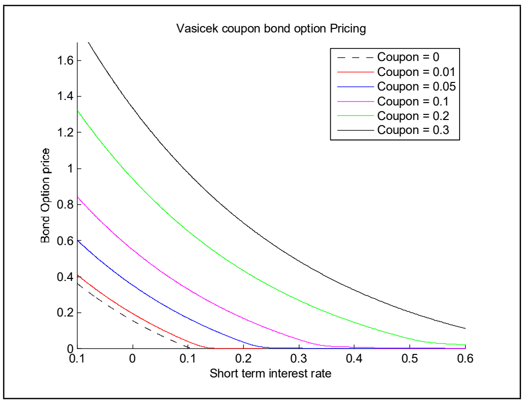

Figure 27: *Comparison of coupon bond options*

3.3 Swaptions

Swaptions are contracts that give the owner the right to enter in a single Swap. If at the option's maturity the swap rate is above the strike, the option is exercised and the owner enters in a swap contract. As highlighted by [14], a long swaption can be employed to take advantage of the current low financing costs to some future uncertain project without being locked into a long-term swap. Another advantage is that swaption is cheaper than interest rate caps, which could be used for the same purposes.

Formally, the yield-based forward-starting payer Swaption for an underlying swap is given as

$$SWPT_Y \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\max\Big(SWAP_Y, 0\Big)|\mathcal{F}_t\right] \quad \forall \ t \in [0, T], \tag{45}$$

where SWAP_Y is calculated as the equivalent representation for a swap contract, exchanging a yield-based floating rate at $\beta - 1$ payment dates paid in-arrears is

$$SWAP_Y = N\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{\beta-1} e^{-\int_t^{T_{i+1}} r_s ds}\left(L(r_T, T_i, T_{i+1}) - K\right)\tau|\mathcal{F}_t\right] \quad (46)$$

$$= N\Big(P(t,T_1) - P(t,T_\beta) - K\sum_{i=1}^{\beta-1}\tau(T_i,T_{i+1})P(t,T_{i+1})\Big). \quad (47)$$

The payoff of the Swaption at time T also can be viewed as

$$SWPT_Y = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\max\Big(L - P(T,S)e^{K\tau(T,S)}, 0\Big)|\mathcal{F}_t\right] \quad \forall\ s \in [t,T]. \quad (48)$$

Then we can follow the model of [8] to calculate the price of the Swaption:

$$SWPT_Y = NP(t,T)\Phi(-d_2) - NP(t,S)e^{K\tau(T,S)}\Phi(-d_1), \quad (49)$$

where

$$d_1 = \frac{ln\left(\frac{P(t,S)e^{K\tau(T,S)}}{P(t,T)}\right) + \frac{\sigma_p^2}{2}}{\sigma_p} \quad (50)$$

$$d_2 = d_1 - \sigma_p \quad (51)$$

and σ_p is given by Eq. (34).

The example shown in Figure 28 stands for a one-year option to enter in a two-year swap and receive 0.13 of the principal at maturity and pay the floating rate. The parameters were set as $a = 0.1$, $b = 0.07$ and $\sigma = 0.015$.

The relative errors shown in Figure 29 are calculated according to Eq. (36).

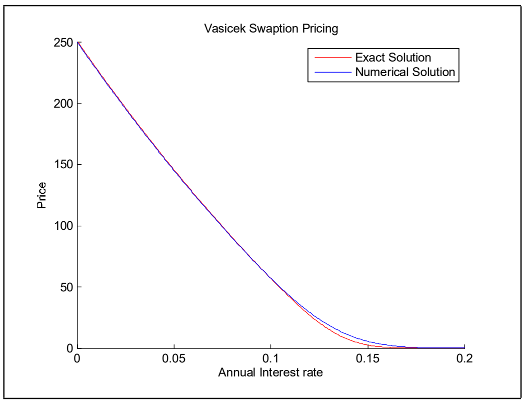

Figure 28: *Swaption pricing*

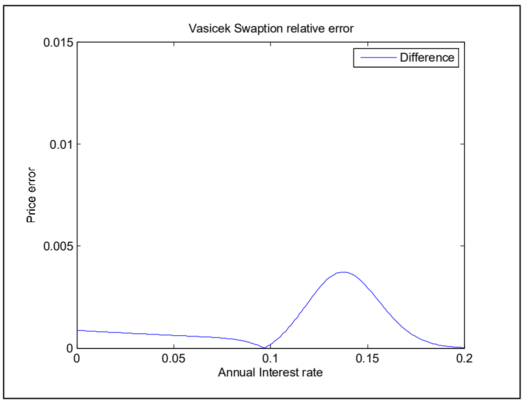

Figure 29: *Swaption error*

3.4 Callable bonds

A callable bond is a bond in which the issuer has the option to redeem it prior to its maturity for a specified price. There are other over-the-counter derivatives, e.g. swaps, with this feature embedded. The calling can be allowed continuously, in an interval of time or in a set of dates, and the notice of the exercising decision is typically, but not necessarily, made in advance.

Issue callable bonds are interesting when the firm expects falling interest rate. Instead of maintaining high coupon payments, the issuer can call the debt back and issue another debt at lower rates. On the other hand the issuer will pay for this flexibility, raising lesser funds than they would do selling straight coupon bonds.

Puttable bonds refer to the opposite, where the owner has the right to require the principal payment earlier. It is interesting to embed this option in the contract when the buyer expects rising interest rates. A bond may contain both callable and puttable options embedded.

To solve numerically a callable bond which contains a discrete set of specific call dates (called semi-American option[3]), we resort to the price updating algorithm described in [5]. The authors solved the callable bond price numerical problem through the Finite Volume method.

Suppose a coupon-bearing bond with M coupons and $N < M$ call dates coinciding with the last N coupon dates and N notice dates. The steps for pricing are the following:
- Solve the coupon-bearing bond with the Modified Full implicit method from bond maturity T to the last notice date τ_{n_M} adding the appropriate coupon at τ_{c_M}. Denote by $K(r, \tau_{n_M}^-)$ the value of the bond immediately before the notice date (backwards in time).
- Denote $X(t_{c_i})$ the call price at time t_{c_i}. This is the value the issuer pays if he/she calls the bond back. Discount it from τ_{c_M} to τ_{n_M} adding the coupon C, or else,

$$[X(t_{c_M}) + C]P(r, \tau_{n_M} - \tau_{c_M}). \tag{52}$$

- Compute the break-even interest rate r_b via

$$[X(t_{c_M}) + C]P(r_b, \tau_{n_M} - \tau_{c_M}) = K(r_b, \tau_{c_M}^-). \tag{53}$$

The break-even interest rate is the one which is not optimal either calling the bond back or maintaining it. It is important to highlight that issuer's optimal decision is to redeem the bond if the value of the callable bond exceeds the present value of the call price plus the coupon; in other words, if the value to refund is greater than the cost to call the bond back.

[3]American options are the class of options that can be exercised anytime during its life. A semi-American contract can be exercised in a predetermined set of dates or periods prior to its maturity.

- In order to preclude arbitrage, update the value of the bond with the following rule (denoted by method 1 in [5]):

$$K(r,\tau_{n_M}^{+}) = \begin{cases} [X(t_{c_M}) + C]P(r, \tau_{n_M} - \tau_{c_M}) & \text{if} r \leq r_b \\ K(r, \tau_{n_M}^{-}) & \text{otherwise,} \end{cases}$$

and solve the coupon bond PDE to the next notice date.
- Repeat it for the remaining call dates to get the price of the callable bond at time zero.

In the case of American callable bonds, i.e., contracts which the issuer can call the bond back anytime prior to maturity, the break-even interest rate is calculated continuously and the value $K(r,\tau_n^+)$ is updated in order to prevent arbitrage opportunities.

We now compare the callable bond prices obtained according to the Modified Full Implicit method and according to [5]. We do this using the example given in [5] itself, along with the numerical results for the prices they presented. The example goes as follows. Suppose that the short-term interest rate follows the Vasicek dynamic with parameters

$$a = 0.44178462, \quad b = 0.0348468515 \quad \text{and} \quad \sigma = 0.13264223.$$

We also assume that the market price of risk is $\lambda = 0.21166329$. We intend to price a callable bond maturing in 20.172 years, paying 4.25% yearly coupons with notice period equal to 0.1666 years. Table 1 lists the 10 call prices relative to the possible call dates (years to maturity). Table 2 presents the comparisons.

It is not clear how many points in the spatial grid the authors used to reach those prices. However, we inferred from some data in their paper that the grid size was 2400 points. So we did the same. In this scenario, the discrepancy between prices in both methods was less than 0.6% for an interest rate range varying from 2% to 20%. Figure 30 exhibits the prices of the callable and straight bonds. As expected, the owner pays less to give the issuer the right to call the bond back. Moreover, the break-even interest rate we obtained was -13.64%, very similar to that of -13.56% found by [5].

By comparing them, we can strongly indicate the good performance of both [5]'s finite volume method and our Modified Full Implicit finite difference method in solving numerically the callable bond price problem.

Table 1: Call dates and prices

Time to maturity	Call Price
1 - 5	1.000
6	1.005
7	1.010
8	1.015
9	1.020
10	1.025

Table 2: Comparison of callable bond pricing with Modified Full Implicit method and the finite volume method developed by [5]

Interest rate	[5]	This work	Absolute error	Relative error
0.02	0.82627	0.83015	0.00388	0.47%
0.03	0.81007	0.81392	0.00385	0.48%
0.04	0.79420	0.79803	0.00383	0.48%
0.05	0.77868	0.78247	0.00379	0.49%
0.06	0.76348	0.76723	0.00375	0.49%
0.07	0.74860	0.75232	0.00372	0.50%
0.08	0.73403	0.73772	0.00369	0.50%
0.09	0.71977	0.72342	0.00365	0.51%
0.10	0.70578	0.70942	0.00364	0.52%
0.11	0.69214	0.69571	0.00357	0.52%
0.12	0.67875	0.68229	0.00354	0.52%
0.13	0.66565	0.66915	0.00350	0.53%
0.14	0.65283	0.65628	0.00345	0.53%
0.15	0.64027	0.64369	0.00342	0.53%
0.16	0.62798	0.63135	0.00337	0.54%
0.17	0.61594	0.61927	0.00333	0.54%
0.18	0.60416	0.60744	0.00328	0.54%
0.19	0.59262	0.59586	0.00324	0.55%
0.20	0.58132	0.58452	0.00320	0.55%

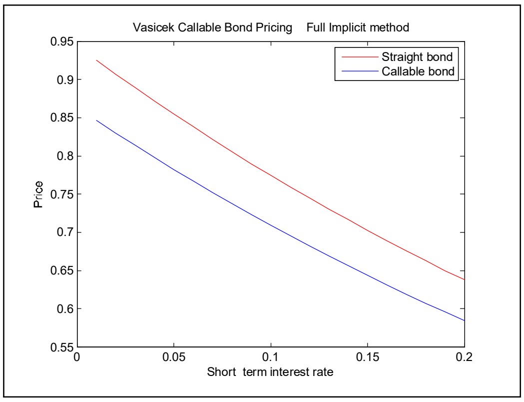

Figure 30: *Callable and Straight bond prices*

3.5 Other interest rate derivatives

In the earlier sections, we have examined the PDE made solutions of some important fixed-income contracts of the financial market which exhibit in the literature closed form pricing formulas or an alternative numerical solution solved via some other methodology (which was the case of the callable bonds). It is impossible to fill up an exhaustive list of interest rate derivatives due to the constant demand of the economic agents for customized products and the speedy progress of financial engineering. However, we shall present some other common instruments and their associated terminal condition.

· Captions and Floortions are like Swaptions. They are compound options in the sense that the owner have an option to enter in another option. A Caption is simply the right to buy or sell an Interest Rate Cap at some time T for a prescribed strike price K with maturity in S. They give more flexibility to the risk manager. Its corresponding PDE terminal condition, i.e., its payoff, is

$$CPT(T) = \omega \max(CAP(T,S) - K, 0), \tag{54}$$

where ω is 1 if the contract gives the right to buy a Cap and -1 otherwise.

In contrast to this, the Floortion offers the owner the right to buy or sell an Interest Rate Floor. Its terminal condition or payoff reads as

$$FLT(T) = \omega \max(Floor(T,S) - K, 0). \tag{55}$$

The strategy here is the same we previously used for pricing bond options and swaptions. We compute the T-price of a cap/floor and feed the above terminal conditions to solve for caption/floortion.

· Range notes are derivatives that pay continuously the interest on a notional principal whenever the interest rate lies between some prescribed lower (r_l) and upper (r_u) bounds. The corresponding PDE for this case differs from Eq. (5) - which adapts to all cases treated up to now - and was devised by [16]. More explicitly, it reads as

$$\frac{\partial U}{\partial t} + a(b-r)\frac{\partial U}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial r^2} + \mathbb{I}(r) = rU \tag{56}$$

with terminal condition

$$U(T,r) = 0, \tag{57}$$

where

$$\mathbb{I}(r) = r \ \text{ if } \ r_l < r < r_u \ \text{ and is zero otherwise.} \tag{58}$$

· Asian options are the class of derivative instruments whose payout is determined by some function of the interest rate. For example, the moving average cap. Its terminal condition is given by

$$U(T,r) = \max\left(\frac{1}{N}\sum_{i=1}^{N} r_i - K, 0\right), \tag{59}$$

If exercised, the contract pays the difference between the mean of the N last observations of the interest rate and the strike K, instead of the last observation of r, as the standard interest rate Cap does. An important example of Asian options will be given ahead. This will be an important result in this work, since we will provide good estimates for pricing a certain Brazilian interest rate derivative.

· Barrier options are contracts that start or cease to exist when a prescribed barrier is reached. For example, an Up-and-Out Cap: this specific contract have terminal conditions expressed as

$$U(T,r) = \max(r(T) - K, 0) \tag{60}$$

and

$$U(t,r) = 0 \ \text{ if } \ r(t) > r_B \ \text{ for some } \ t. \tag{61}$$

The interest rate Cap ceases to exist whenever $r(t)$ touches the barrier r_B and pays $\max(r(T) - K, 0)$ otherwise.

· American options are those that can be exercised at anytime during the life of the contract. This optimal stopping problem of finding the time to exercise the option, results in the following PDE problem

$$\frac{\partial U}{\partial t} + a(b-r)\frac{\partial U}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial r^2} - rU \leq 0, \tag{62}$$

with the terminal condition

$$U(T, r) = h(r) \tag{63}$$

and the no-arbitrage constraint

$$U(t, r) \geq h(r), \tag{64}$$

where $h(r)$ denotes the contract payoff.

It means that the holder of the option controls the early exercise feature [16] and the writer can earn more than the risk-free rate if the option was not optimally exercised. Since American options give its owner additional rights, they probably have higher prices. This difference in prices are called the early exercise premium.

Swaptions, Bond options, Captions and many other derivatives can have early exercise features. The class of derivatives which the decision dates is specified as a set of discrete days or intervals is known as Bermudan Options, like the Callable bonds treated above.

In order to exhaust our list of numerical pricing problems and attest the good performance of the Modified Full Implicit method, we engineered a Digital Bond Option pricing. It is a derivative that pays

$$U(T, r) = \begin{cases} 1 & \text{if} P(T, S) - K > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It means that the holder receives 1 if the bond price exceeds the strike, and zero otherwise.

So, we priced a digital option which pays 1 if the price of a 4-year zero-coupon bond exceeds the strike $= 0.82$ in two years. The parameters are set as $a = 0.2$, $b = 0.1$ and $\sigma = 0.02$. The solution is showed in Figure 32.

It is interesting noticing that for deep in-the-money options the solution presents an identity function behavior. This fact is due to the high probability of the option paying 1 at maturity, which means the holder is long in nothing but the zero-coupon bond itself.

We created another contract aiming to increase the number of discontinuities in the pay-off. It is a double digital option which pays

$$U(T, r) = \begin{cases} 1 & \text{if} P(T, S) - K > 0 \\ 1 & \text{if} P(T, S) - K < -0.15 \\ 0 & \text{otherwise.} \end{cases}$$

The payoff and the option prices are illustrated in Figures 33 and 34, respectively. The same parameters were preserved for this case. Figure 35 shows the solution surface of the EDP. We notice that it is smooth (except very near to the payoff, as it ought to be). At

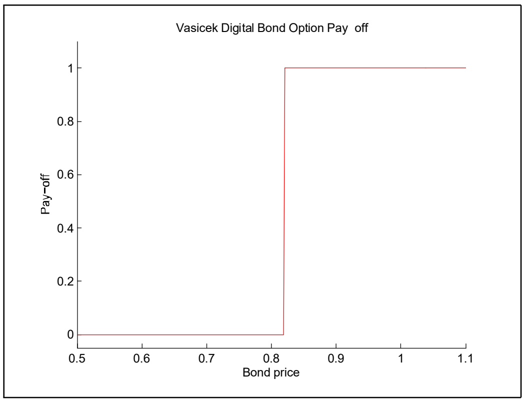

Figure 31: *Digital Bond Option pay-off*

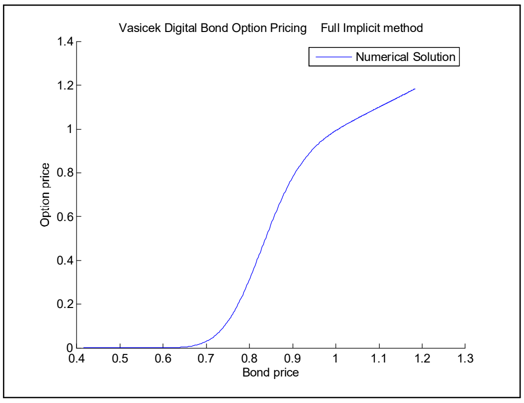

Figure 32: *Digital Bond Option prices*

this point, an important notice is to quote [6]: "It is well known that discontinuous initial conditions adversely impact the accuracy of finite difference schemes. In particular, the solution of the difference schemes exhibits oscillations just after $t = 0$." This is exactly what does not occur in the examples here, where the Modified Full Implicit (finite difference) scheme is applied: the pricing solution on both examples is smooth and remains free of the undesired spurious oscillations despite the presence of an abrupt discontinuity in the terminal condition (see Figure 31).

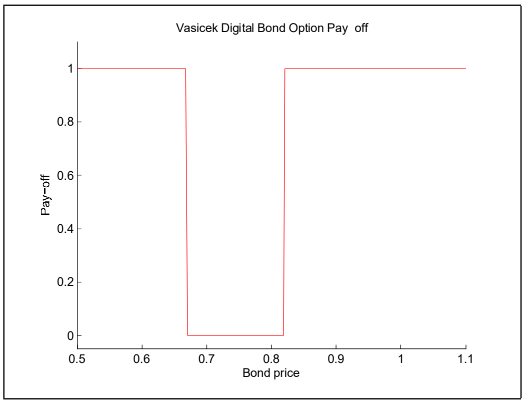

Figure 33: *Double Digital Bond Option pay-off*

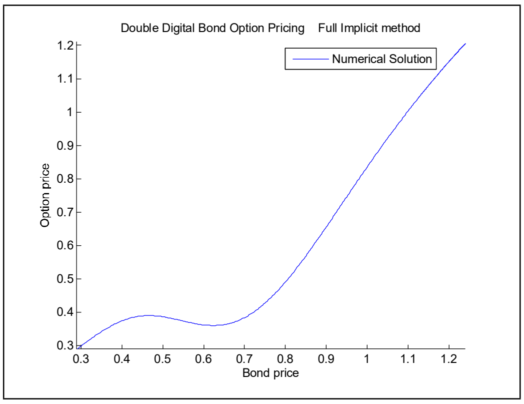

Figure 34: *Double Digital Bond Option prices*

4 CONCLUSIONS

We benefited from the good results that the numerical method gives to price derivatives in the fixed-income market. We model the most common fixed-income contracts and provided numerical examples, comparing them with their closed-form solutions - when they exist. It is interesting that the Feynman-Kac procedure favors the engineering of a large class of financial instruments. The resulting PDEs can accommodate

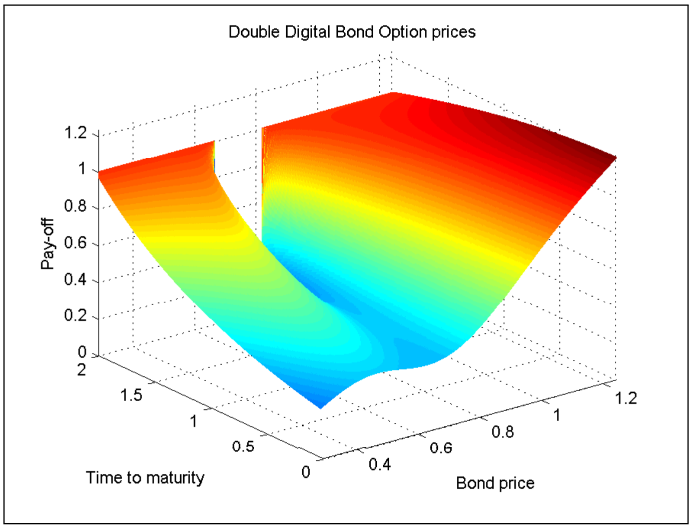

Figure 35: *Double Digital Bond Option prices surface*

uncommon features of the contract, such as variable coupon rates, without compromise the computational effort. If one chooses another interest rate dynamic - as opposed to Vasicek, the results can be easily extended.

Acknowledgments

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APPENDIX A

Discounted Feynman-Kac Theorem [12]:
Let $T > 0$, $h : \mathbb{R} \to \mathbb{R}$ and

$$U(t,r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} h(r_T)|\mathcal{F}_t\right], \tag{65}$$

where r is the solution of the SDE Eq. (1) with initial condition $r = r_t$. Then $U(t,r)$ solves

$$\frac{\partial U}{\partial t} + a(b-r)\frac{\partial U}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial r^2} = rU \tag{66}$$

with terminal condition

$$U(T,r) = h(r). \tag{67}$$